Fast Convergence of Natural Bargaining Dynamics in Exchange Networks

Yashodhan Kanoria* Mohsen Bayati* Christian Borgs† Jennifer Chayes†
Andrea Montanari*,‡

October 21, 2010

Abstract

Bargaining networks model the behavior of a set of players who need to reach pairwise agreements for making profits. Nash bargaining solutions in this context correspond to solutions which are stable and balanced. Kleinberg and Tardos [19] proved that, if such solutions exist, then they can by calculated in polynomial time. This left open the question: Are there dynamics which can describe the bargaining process of real-world players, and which converge quickly to a Nash bargaining solution? This paper provides an affirmative answer to that question.

The contribution of this paper is threefold: (1) We introduce a single-stage local dynamics which models the way in which actual players could bargain. We show that (approximate) fixed points of our dynamics are in one-to-one correspondence with (approximate) Nash bargaining solutions. (2) We prove that our dynamics converges to an ϵ-fixed point in $O(1/\epsilon^2)$ iterations independent of the network size when the potential earnings (weights) are uniformly bounded. We use this to prove that an approximate Nash bargaining solution is reached in time polynomial in $1/\epsilon$, the network size and $1/g$. Here g is the difference between the weights of the two corners of the matching polytope having largest weights, and controls the behavior of fast message passing algorithms for maximum weight matching (matching naturally arises as a subproblem of Nash bargaining). (3) Our proof introduces a new powerful technique from functional analysis to this set of problems. The technique allows us to extend our results in various directions. We believe the tools introduced here will be useful in many related problems.

As a corollary, for bipartite graphs we prove polynomial time convergence to an approximate Nash bargaining solution, with probability close to one under small random perturbations.

1 Introduction and main results

Exchange networks model social and economic relations among individuals under the premise that any relationship has a potential value for its partners. In a purely economic setting, one can imagine that each relation corresponds to a trading opportunity, and its value is the amount of money to be earned from the trade. A fascinating question in this context is that of how network structure influences the power balance between nodes (i.e. their earnings).

Controlled experiments [32, 21, 29] have been carried out by sociologists in a set-up that can be summarized as follows. A graph $G = (V, E)$ is defined, with positive weights $w_{ij} > 0$ associated to the edges $(i, j) \in E$. A player sits at each node of this network, and two players connected by edge (i, j) can share a profit of w_{ij} dollars if they agree to trade with each other. Each player can trade with at most one of her neighbors (this is called the 1-*exchange rule*), so that a set of valid trading pairs forms a matching M in the graph G. It is often the case that players are provided information only about their immediate neighbors.

Network exchange theory studies the possible outcomes of such a process. While each instance admits a multitude of outcomes, special classes of outcomes are selected on the basis of 'desirable' properties. In this paper, we focus on 'balanced outcomes', a solution concept that dates back to Nash's bargaining theory [23], and was generalized in [24, 10, 19]. In balanced outcomes, earnings are such that each transaction follows the Nash bargaining solution given the earnings in the rest of the network. Alternative solution concepts for bargaining on networks were studied in [9].

We define an *outcome* or *trade outcome* as a pair $(M, \underline{\gamma})$ where $M \subseteq E$ is a matching of G, and $\underline{\gamma} = \{\gamma_i : i \in V\}$ is the vector of players' profits. This means, $\gamma_i \geq 0$, and $(i, j) \in M$ implies $\gamma_i + \gamma_j = w_{ij}$, whereas

*Department of Electrical Engineering, Stanford University
†Microsoft Research New England
‡Department of Statistics, Stanford University

for every unmatched node $i \notin M$ we have $\gamma_i = 0$.

A balanced outcome, or Nash bargaining (NB) solution, is a trade outcome that satisfies the additional requirements of *stability* and *balance*. Denote by ∂i the set of neighbors of node i in G.

Stability. If player i is trading with j, then she cannot earn more by simply changing her trading partner. Formally $\gamma_i + \gamma_j \geq w_{ij}$ for all $(i, j) \in E \setminus M$.

Balance. If player i is trading with j, then the surplus of i over her best alternative must be equal to the surplus of j over his best alternative. Mathematically,

$$\gamma_i - \max_{k \in \partial i \setminus j} (w_{ik} - \gamma_k)_+ = \gamma_j - \max_{l \in \partial j \setminus i} (w_{jl} - \gamma_l)_+ \tag{1.1}$$

for all $(i, j) \in M$.

It turns out that the interplay between the 1-exchange rule and the stability and balance conditions results in highly non-trivial predictions regarding the influence of network structure on individual earnings. Some of these predictions agree with experimental findings, but alternative predictive frameworks exist as well [29].

1.1 Related work and our contribution Recall the LP relaxation to the maximum weight matching problem

$$\begin{aligned}
(1.2) \qquad & \text{maximize} & & \sum_{(i,j) \in E} w_{ij} x_{ij}, \\
& \text{subject to} & & \sum_{j \in \partial i} x_{ij} \leq 1 \quad \forall i \in V, \\
& & & x_{ij} \geq 0 \quad \forall (i, j) \in E\,.
\end{aligned}$$

The dual problem to (1.2) is

$$\begin{aligned}
(1.3) \qquad & \text{minimize} & & \sum_{i \in V} y_i, \\
& \text{subject to} & & y_i + y_j \geq w_{ij} \quad \forall (i, j) \in E, \\
& & & y_i \geq 0 \quad \forall i \in V.
\end{aligned}$$

Stable outcomes were studied by Sotomayor [28].

Proposition 1.1. [28] *Stable outcomes exist if and only if the linear programming relaxation* (1.2) *of the maximum weight matching problem on G admits an integral optimum. Further, if $(M, \underline{\gamma})$ is a stable solution then M is a maximum weight matching and $\underline{\gamma}$ is an optimum solution to the dual LP* (1.3).

Following [24, 10], Kleinberg and Tardos [19] first considered balanced outcomes on general exchange networks and proved that: *a network G admits a balanced outcome if and only if it admits a stable outcome.*

The same paper describes a polynomial algorithm for constructing balanced outcomes. This is in turn based on the dynamic programming algorithm of Aspvall and Shiloach [1] for solving systems of linear inequalities. However, [19] left open the question of how the actual bargaining process converges to balanced outcomes.

Azar and co-authors [2] first studied the question as to whether a balanced outcome can be produced by a local dynamics, and were able to answer it positively. Their results left, however, two outstanding challenges: (I) The bound on the convergence time proved in [2] is exponential in the network size, and therefore does not provide a solid justification for convergence to NB solutions in large networks; (II) The algorithm analyzed by these authors first selects a matching M in G using the message passing algorithm studied in [5, 16, 6, 26], corresponding to the pairing of players that trade. In a second phase the algorithm determines the profit of each player. While such an algorithm can be implemented in a distributed way, Azar et al. point out that it is not entirely realistic. Indeed the rules of the dynamics change abruptly after the matching is found. Further, if the pairing is established at the outset, the players lose their bargaining power.

The present paper aims at tackling these challenges. First we introduce a natural dynamics that is interpretable as a realistic negotiation process. We show that the fixed points of the dynamics are in one to one correspondence with NB solutions, and prove that it converges to such solutions. Moreover, we show that the convergence to approximate NB solutions is fast. Furthermore we are able to treat the more general case of nodes with unsymmetrical bargaining powers and generalize the result of [19] on existence of NB solutions to this context. These results are obtained through a new and seemingly general analysis method, that builds on powerful quantitative estimates on mappings in the Banach spaces [3]. For instance, our approach allows us to prove that a simple variant of the edge balancing dynamics of [2] converges in polynomial time (see Appendix A, esp. Section A.2).

Natural dynamics and its analysis have similarities with a series of papers on using max-product belief propagation for the weighted matching problems [5, 16, 6, 26]. We discuss that connection and extensions of our results to those settings in Appendix F.

Rochford [24], and recent work by Bateni et al [4], relate the exchange networks problem to the extensive literature on cooperative game theory. A consequence of the connection established is that the results of Kleinberg and Tardos [19] and Azar et al [2] are implied by previous work in the economics literature. However, [24, 4] also leave open the twin questions of finding (i) a fast local dynamics, and (ii) a natural model for bargaining. Another work worth mention is by Faigle et al [13]: it provides a polynomial time algorithm for finding balanced outcomes (in a more general setting). The algorithm involves local 'transfers', alternating with a *non-local* LP based step after every $O(n^2)$ transfers.

In a parallel work by one of us [17], it is shown that the natural dynamics may take exponentially long to converge in the case of unequal bargaining powers. Other algorithms like that of Kleinberg and Tardos [19] fail to generalize. However, a suitable modification to the bargaining process is shown to be an FPTAS even for unequal bargaining powers.

1.2 A natural dynamics It is a fundamental open question whether NB solutions describe the outcomes of actual bargaining processes. The stream of controlled experiments on small networks will surely help to get an answer [32, 8]. [8], in particular, provides supportive evidence while also indicating the presence of some unmodeled effects. On the other hand, an important step forward in our theoretical understanding was achieved by Kleinberg and Tardos [19] who proved that NB solutions can be constructed in polynomial time.

However, even a superficial look at experimental conditions, e.g. in [8], reveals that players cannot possibly run the algorithm described in [19]. The algorithm requires the solution of a sequence of linear programs (involving global information), that successively fix node earnings. There are two possibilities: Either there exists a realistic model for the bargaining dynamics that converges to NB solutions, or the solution concept has to be revised. For the former possibility, the underlying dynamics should satisfy the following requirements: (1) It should converge rapidly to NB solutions; (2) It should be *natural*.

While the first requirement is easy to define and motivate, the second one is more subtle but not less important. A few properties of a natural dynamics are the following ones: It should be *local*, i.e. involve limited information exchange along edges and processing at nodes; It should be *time invariant*, i.e. the players' behavior should be the same/similar on identical local information at different times; It should be *interpretable*, i.e. the information exchanged along the edges should have a meaning for the players involved, and should be consistent with reasonable behavior for players.

In the model we propose, at each time t, each player sends a message to each of her neighbors. The message has the meaning of 'best current alternative'. We denote the message from player i to player j by $\alpha^t_{i\backslash j}$. Player i is telling player j that she (player i) currently estimates earnings of $\alpha^t_{i\backslash j}$ elsewhere, if she chooses not to trade with j.

The vector of all such messages is denoted by $\underline{\alpha}^t \in \mathbb{R}_+^{2|E|}$. Each agent i makes an 'offer' to each of her neighbors, based on her own 'best alternative' and that of her neighbor. The offer from node i to j is denoted by $m^t_{i\to j}$ and is computed according to

$$m^t_{i\to j} = (w_{ij} - \alpha^t_{i\backslash j})_+ - \frac{1}{2}(w_{ij} - \alpha^t_{i\backslash j} - \alpha^t_{j\backslash i})_+ \,. \tag{1.4}$$

It is easy to deduce that this definition corresponds to the following policy: (i) An offer is always nonnegative, and a positive offer is never larger than $w_{ij} - \alpha^t_{i\backslash j}$ (no player is interested in earning less than her current best alternative); (ii) Subject to the above constraints, the surplus $(w_{ij} - \alpha^t_{i\backslash j} - \alpha^t_{j\backslash i})$ (if nonnegative) is shared equally. We denote by $\underline{m}^t \in \mathbb{R}_+^{2|E|}$ the vector of offers.

Notice that $\underline{m}^t$ is just a deterministic function of $\underline{\alpha}^t$. In the rest of the paper we shall describe the network status uniquely through the latter vector, and use $\underline{m}|_{\underline{\alpha}^t}$ to denote $\underline{m}^t$ defined by (1.4) when required so as to avoid ambiguity.

Each node can estimate its potential earning based on the network status, using

$$\gamma^t_i \equiv \max_{k\in\partial i} m^t_{k\to i}, \tag{1.5}$$

the corresponding vector being denoted by $\underline{\gamma}^t \in \mathbb{R}_+^{|V|}$. Notice that $\underline{\gamma}^t$ is also a function of $\underline{\alpha}^t$.

Messages are updated synchronously through the network, according to the rule

$$\alpha^{t+1}_{i\backslash j} = (1-\kappa)\,\alpha^t_{i\backslash j} + \kappa \max_{k\in\partial i\backslash j} m^t_{k\to i}\,. \tag{1.6}$$

Here $\kappa \in (0, 1)$ is a 'damping' factor: $(1 - \kappa)$ can be thought of as the inertia on the part of the nodes to update their current estimates (represented by outgoing messages). The use of $\kappa < 1$ eliminates pathological behaviors related to synchronous updates. In particular, we observe oscillations on even-length cycles in the undamped synchronous version. We mention here that in Appendix B we present extensions of our results to various update schemes (e.g., asynchronous updates, time-varying damping factor).

Remark 1.1. *An update under the natural dynamics requires $O(|E|)$ operations in total.*

Let $W \equiv \max_{(ij)\in E} w_{ij}$. Often in the paper we take $W = 1$, since this can always be achieved by rescaling the problem, which is the same as changing units. It is easy to see that $\underline{\alpha}^t \in [0, W]^{2|E|}$, $\underline{m}^t \in [0, W]^{2|E|}$ and $\underline{\gamma}^t \in [0, W]^{|V|}$ at all times (unless the initial condition violates this bounds). Thus we call $\underline{\alpha}$ a 'valid' message vector if $\underline{\alpha} \in [0, W]^{2|E|}$.

1.3 Main results: Fixed point properties and convergence Our first result is that fixed points of the update equations (1.4), (1.6) (hereafter referred to as 'natural dynamics') are indeed in correspondence with Nash bargaining solutions when such solutions exist. *Note that the fixed points are independent of the damping factor κ.* The correspondence with NB solutions includes pairing between nodes, according to the following notion of induced matching.

DEFINITION 1.1. *We say that a state $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ (or just $\underline{\alpha}$)* induces a matching M *if the following happens. For each node $i \in V$ receiving non-zero offers ($m_{\cdot\to i} > 0$), i is matched under M and gets its unique best offer from node j such that $(i,j) \in M$. Further, if $\gamma_i = 0$ then i is not matched in M. In other words, pairs in M receive unique best offers that are positive from their respective matched neighbors whereas unmatched nodes receive no non-zero offers.*

Consider the LP relaxation to the maximum weight matching problem (1.2). A feasible point $\underline{x}$ for LP (1.2) is called *half-integral* if for all $e \in E$, $x_e \in \{0, 1, \frac{1}{2}\}$. It is well known that problem (1.2) always has an optimum $\underline{x}^*$ that is half-integral [27]. An LP with a fully integer $\underline{x}^*$ ($x^*_e \in \{0,1\}$) is called *tight*.

THEOREM 1.1. *Let G be an instance admitting one or more Nash bargaining solutions, i.e. the LP (1.2) admits an integral optimum.*
(a) Unique LP optimum (generic case): *Suppose the optimum is unique corresponding to matching M^*. Let $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ be a fixed point of the natural dynamics. Then $\underline{\alpha}$ induces matching M^* and $(M^*, \underline{\gamma})$ is a Nash bargaining solution. Conversely, every Nash bargaining solution $(M', \underline{\gamma}_{\rm NB})$ has $M' = M^*$ and corresponds to a unique fixed point of the natural dynamics with $\underline{\gamma} = \underline{\gamma}_{\rm NB}$.*

(b) Let $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ be a fixed point of the natural dynamics. Then $(M^*, \underline{\gamma})$ is a Nash bargaining solution for any integral maximum weight matching M^*. Conversely, if $(M', \underline{\gamma}_{\rm NB})$ is a Nash bargaining solution, M' is a maximum weight matching and there is a unique fixed point of the natural dynamics with $\underline{\gamma} = \underline{\gamma}_{\rm NB}$.

We prove Theorem 1.1 in Section 3. Theorem D.1 in Appendix D extends this characterization of fixed points of the natural dynamics to cases where Nash bargaining solutions do not exist.

REMARK 1.2. *The condition that a tight LP* (1.2) *has a unique optimum is generic (see Appendix D, Remark D.1). Hence, fixed points induce a matching for almost all instances (cf. Theorem 1.1(a)). Further, in the non-unique optimum case, we cannot expect an induced matching, since there is always some node with two equally good alternatives.*

The existence of a fixed point of the natural dynamics is immediate from Brouwer's fixed point theorem. Our next result says that the natural dynamics always converges to a fixed point.

THEOREM 1.2. *The natural dynamics has at least one fixed point. Moreover, for any initial condition with $\underline{\alpha}^0 \in [0, W]^{2|E|}$, $\underline{\alpha}^t$ converges to a fixed point.*

Note that Theorem 1.2 does not require any condition on LP (1.2). It also does not require uniqueness of the fixed point.

The proof is in Section 2.

With Theorems 1.1 and 1.2, we know that in the limit of a large number of iterations, the natural dynamics yields a Nash bargaining solution. However, this still leaves unanswered the question of the rate of convergence of the natural dynamics. Our next theorem addresses this question, establishing fast convergence to an approximate fixed point.

However, before stating the theorem we define the notion of approximate fixed point.

DEFINITION 1.2. *We say that $\underline{\alpha}$ is an ϵ*-fixed point*, or ϵ*-FP *in short, if, for all $(i,j) \in E$ we have*

$$\left|\alpha_{i\backslash j} - \max_{k \in \partial i \backslash j} m_{k\to i}\right| \le \epsilon\,, \tag{1.7}$$

and similarly for $\alpha_{j\backslash i}$. Here, $\underline{m}$ is obtained from $\underline{\alpha}$ through Eq. (1.4) *(i.e., $\underline{m} = \underline{m}|_{\underline{\alpha}}$).*

Note that ϵ-fixed points are also defined independently of the damping κ.

THEOREM 1.3. *Let $G = (V, E)$ be an instance with weights $(w_e, e \in E) \in [0,1]^{|E|}$. Take any initial condition $\underline{\alpha}^0 \in [0,1]^{2|E|}$. Take any $\epsilon > 0$. Define*

$$T^*(\epsilon) = \frac{1}{\pi\kappa(1-\kappa)\epsilon^2}\,. \tag{1.8}$$

Then for all $t \ge T^*(\epsilon)$, $\underline{\alpha}^t$ is an ϵ-fixed point. (Here $\pi = 3.14159\ldots$)

Thus, if we wait until time t, we are guaranteed to obtain an $\left(1/\sqrt{\pi\kappa(1-\kappa)t}\right)$-FP. Theorem 1.3 is proved in Section 2.

REMARK 1.3. *For any $\epsilon > 0$, it is possible to construct an example such that it takes $\Omega(1/\epsilon)$ iterations to reach an ϵ-fixed point. This lower bound can be improved to $\Omega(1/\epsilon^2)$ in the unequal bargaining powers case (cf. Section 5). However, in our constructions, the size of the example graph grows with decreasing ϵ in each case.*

We are left with the problem of relating approximate fixed points to approximate Nash bargaining solutions. We use the following definition of ϵ-Nash bargaining solution, that is analogous to the standard definition of ϵ-Nash equilibrium (e.g., see [11]).

DEFINITION 1.3. *We say that* $(M, \underline{\gamma})$ *is an* ϵ-Nash bargaining solution *if it is a valid trade outcome that is stable and satisfies* ϵ*-balance.* ϵ*-balance means that for every* $(i, j) \in M$ *we have*

(1.9)
$$\left|[\gamma_i - \max_{k\in\partial i\backslash j}(w_{ik}-\gamma_k)_+] - [\gamma_j - \max_{l\in\partial j\backslash i}(w_{jl}-\gamma_l)_+]\right| \leq \epsilon\,.$$

A subtle issue needs to be addressed. For an approximate fixed point to yield an approximate Nash bargaining solution, a suitable pairing between nodes is needed. Note that our dynamics does not force a pairing between the nodes. Instead, a pairing should emerge quickly from the dynamics. In other words, nodes on the graph should be able to identify their trading partners from the messages being exchanged. As before, we use the notion of an induced matching (see Definition 1.1).

DEFINITION 1.4. *Consider LP* (1.2). *Let* $\mathcal{H}$ *be the set of half integral points in the primal polytope. Let* $\underline{x}^* \in \mathcal{H}$ *be an optimum. Then the LP gap* g *is defined as* $g = \min_{\underline{x}\in\mathcal{H}\backslash\{\underline{x}^*\}} \sum_{e\in E} w_e x_e^* - \sum_{e\in E} w_e x_e$.

THEOREM 1.4. *Let* G *be an instance for which the LP (1.2) admits a unique optimum, and this is integral, corresponding to matching* M^**. Let the gap be* $g > 0$*. Let* $\underline{\alpha}$ *be an* ϵ*-fixed point of the natural dynamics, for some* $\epsilon < g/(6n^2)$*. Let* $\underline{\gamma}$ *be the corresponding earnings estimates. Then* $\underline{\alpha}$ *induces the matching* M^* *and* $(\underline{\gamma}, M^*)$ *is an* (6ϵ)*-Nash bargaining solution. Conversely, every* ϵ*-Nash bargaining solution* $(M', \underline{\gamma}_{\rm NB})$ *has* $M' = M^*$ *for any* $\epsilon > 0$.

Note that $g > 0$ is equivalent to the unique optimum condition (cf. Remarks 2, 5). The proof of this theorem requires generalization of the analysis used to prove Theorem 1.1 to the case of approximate fixed points. Since its proof is similar to the proof of Theorem 1.1, we defer it to Appendix E. We stress, however, that Theorem 1.4 is not, in any sense, an obvious strengthening of Theorem 1.1. In fact, this is a delicate property of approximate fixed points that holds *only in the case of balanced outcomes*. This characterization breaks down in the face of a seemingly benign generalization to unequal bargaining powers (cf. Section 5 and [17, Section 4]).

Theorem 1.4 holds for all graphs, and is, in a sense, the best result we can hope for. To see this, consider the following immediate corollary of Theorems 1.3 and 1.4.

COROLLARY 1.1. *Let* $G = (V, E)$ *be an instance with weights* $(w_e, e \in E) \in [0, 1]^{|E|}$*. Suppose LP (1.2) admits a unique optimum, and this is integral, corresponding to matching* M^**. Let the gap be* $g > 0$*. Then for any* $\underline{\alpha}^0 \in [0, 1]^{2|E|}$*, there exists* $T^* = O(n^4/g^2)$ *such that for any* $t \geq T^*$*,* $\underline{\alpha}^t$ *induces the matching* M^* *and* $(\underline{\gamma}^t, M^*)$ *is an* $(6/\sqrt{\pi\kappa(1-\kappa)t})$*-NB solution.*

Proof. Choose T^* as $T^*(g/(10n^2))$ as defined in (1.8). Clearly, $T^* = O(n^4/g^2)$. From Theorem 1.3, $\underline{\alpha}^t$ is an $\epsilon(t)$-FP for $\epsilon(t) = 1/\sqrt{\pi\kappa(1-\kappa)t}$. Moreover, for all $t \geq T^*$, $\epsilon(t) \leq g/(10n^2)$. Hence, by Theorem 1.4, $\underline{\alpha}^t$ induces the matching M^* and $(\underline{\gamma}^t, M^*)$ is a $(6\epsilon(t))$-NB solution for all $t \geq T^*$.

Corollary 1.1 implies that for any $\epsilon > 0$, the natural dynamics finds an ϵ-NB solution in time $O\left(\max\left(n^4/g^2, 1/\epsilon^2\right)\right)$.

This result is the essentially the strongest bound we can hope for in the following sense. First, note that we need to find M^* (see converse in Theorem 1.4) *and* balance the allocations. Max product belief propagation, a standard local algorithm for computing the maximum weight matching, requires $O(n/g)$ iterations to converge, and *this bound is tight* [6]. Similar results hold for the Auction algorithm [7] which also locally computes M^*. Moreover, max product BP and the natural dynamics are intimately related (see Section F), with the exception that *max product is designed to find* M^**, but this is not true for the natural dynamics.* Corollary 1.1 shows that natural dynamics only requires a time that is polynomial in the same parameters n and $1/g$ to find M^*, while it simultaneously takes rapid care of balancing the outcome! This is very encouraging.

1.3.1 Example: Polynomial convergence to ϵ-NB solution on bipartite graphs.

The next result further shows a concrete setting in which Corollary 1.1 leads to a strong guarantee on quickly reaching an approximate NB solution.

THEOREM 1.5. *Let* $G = (V, E)$ *be a bipartite graph with weights* $(w_e, e \in E) \in [0, 1]^{|E|}$*. Take any* $\xi \in (0, 1), \eta \in (0, 1)$*. Construct a perturbed problem instance with weights* $\bar{w}_e = w_e + \eta U_e$*, where* U_e *are independent identically distributed random variables uniform in* $[0, 1]$*. Then there exists* $C = C(\kappa) < \infty$*, such that for*

(1.10)
$$T^* = C\left(\frac{n^2|E|}{\eta\xi}\right)^2,$$

the following happens for all $t \geq T^*$ *with probability at least* $1 - \xi$*. State* $\underline{\alpha}^t$ *induces a matching* M *that is independent of* t*. Further,* $(\underline{\gamma}^t, M)$ *is a* $\epsilon(t)$*-NB solution for the perturbed problem, with* $\epsilon(t) = 12/\sqrt{\pi\kappa(1-\kappa)t}$.

ξ represents our target in the probability that a pairing does not emerge, while η represents the size of perturbation of the problem instance.

Theorem 1.5 implies that for any fixed η and ξ, and any $\epsilon > 0$, we find an ϵ-NB solution in time $\tau(\epsilon) = K \max(n^4|E|^2, 1/\epsilon^2)$ with probability at least $1-\xi$, where $K = K(\eta, \xi, \kappa) < \infty$. Theorem 1.5 is proved in Section 4.

1.3.2 Other results A different analysis allows us to prove *exponentially fast convergence* to a unique Nash bargaining solution. We describe this briefly in Section 2.1, referring to an earlier version of this paper [18] for the proof. Second, we generalize to the case of *nodes with unsymmetrical bargaining powers*. We show that generalizations of the Theorems 1.1, 1.2 and 1.3 hold for a suitably modified dynamics. This is described in Section 5. Third, if only a fast local algorithm (as opposed to a natural dynamics) for the Nash bargaining problem is desired: we provide a local, poly$(|V|, 1/\epsilon)$ 'reduction' from the problem of finding an ϵ-Nash bargaining solution to the maximum weight matching problem (see Appendix A, Theorem A.1)).

1.4 Outline of the paper In Section 2, we prove Theorems 1.2 and 1.3 regarding convergence of the natural dynamics. We characterize fixed points in Section 3 with a proof of Theorem 1.1 (the proof of Theorem 1.4 is deferred to Appendix E). Section 4 shows polynomial time convergence on bipartite graphs (proof of Theorem 1.5).

In Section 5 we extend some of our results to the unsymmetrical case with nodes having different bargaining powers. As a byproduct, we characterize the existence of solutions in this more general setting. Appendix A shows that given a maximum weight matching, an ϵ-Nash bargaining solution can be constructed in time poly$(|V|, 1/\epsilon)$. Appendix B contains a discussion on variations of the natural dynamics including time and node varying damping factors and asynchronous updates. We relate our dynamics with belief propagation for maximum weight matching in Appendix F.

2 Convergence to fixed points: Proofs of Theorems 1.2 and 1.3

Theorems 1.2 and 1.3 admit a surprisingly simple proofs, that build on powerful results in the theory of non-expansive mappings in Banach spaces.

DEFINITION 2.1. *Given a normed linear space L, and a bounded domain $D \subseteq L$, a* non-expansive mapping *$\mathsf{T} : D \to L$ is a mapping satisfying $\|\mathsf{T}x - \mathsf{T}y\| \le \|x - y\|$ for all $x, y \in D$.*

Mann [22] first considered the iteration $x^{t+1} = (1-\kappa)\, x^t + \kappa\, \mathsf{T}x^t$ for $\kappa \in (0,1)$, which is equivalent to iterating $\mathsf{T}_\kappa = (1-\kappa)\, I + \kappa\, \mathsf{T}$. Ishikawa [14] and Edelstein-O'Brien [12] proved the surprising result that, if the sequence $\{x^t\}_{t\ge 0}$ is bounded, then $\|\mathsf{T}x^t - x^t\| \to 0$ (the sequence is *asymptotically regular*) and indeed $x^t \to x^*$ with x^* a fixed point of T.

Baillon and Bruck [3] recently proved a powerful quantitative version of Ishikawa's theorem: If $\|x^0 - x^t\| \le 1$ for all t, then

$$\|\mathsf{T}x^t - x^t\| < \frac{1}{\sqrt{\pi\kappa(1-\kappa)t}}\,. \tag{2.11}$$

The surprise is that such a result holds irrespective of the mapping T and of the normed space (in particular, of its dimensions). Theorems 1.2 and 1.3 immediately follow from this theory once we recognize that the natural dynamics can be cast into the form of a Mann iteration for a mapping which is non-expansive with respect to a suitably defined norm.

Let us stress that the nonexpansivity property does not appear to be a lucky mathematical accident, but rather an intrinsic property of bargaining models under the one-exchange constraint. It loosely corresponds to the basic observation that if earnings in the neighborhood of a pair of trade partners change by amounts $N_1, N_2, ..., N_k$, then the balanced split for the partners changes at most by $\max(N_1, N_2, \dots, N_k)$, i.e., the largest of the neighboring changes.

Our technique seems therefore applicable in a broader context. (For instance, it can be applied successfully to prove fast convergence of a synchronous and damped version of the edge-balancing dynamics of [2].)

Proof. [Proof (Theorem 1.2)] We consider the linear space $L = \mathbb{R}^{2|E|}$ indexed by directed edges in G. On the bounded domain $D = [0, W]^{2|E|}$ we define the mapping $\mathsf{T} : \underline{\alpha} \mapsto \mathsf{T}\underline{\alpha}$ by letting, for $(i,j) \in E$,

$$(\mathsf{T}\underline{\alpha})_{i\backslash j} \equiv \max_{k \in \partial i \backslash j} m_{k\to i}|_{\underline{\alpha}}\,, \tag{2.12}$$

where $m_{k\to i}|_{\underline{\alpha}}$ is defined by Eq. (1.4). It is easy to check that the sequence of best alternatives produced by the natural dynamics corresponds to the Mann iteration $\underline{\alpha}^t = \mathsf{T}_\kappa^t \underline{\alpha}^0$. Also, T is non-expansive for the ℓ_∞ norm

$$\|\underline{\alpha} - \underline{\beta}\|_\infty = \max_{(i,j)\in E} |\alpha_{i\backslash j} - \beta_{i\backslash j}|\,. \tag{2.13}$$

Nonexpansivity follows from:
(i) The 'max' in Eq. (2.12) is non expansive.
(ii) An offer $m_{i\to j}$ as defined by Eq. (1.4) is non-expansive. To see this, note that $m_{i\to j} = f(\alpha_{i\backslash j}, \alpha_{j\backslash i})$, where $f(x,y) : \mathbb{R}_+^2 \to \mathbb{R}_+$ is given by

$$f(x,y) = \begin{cases} \frac{w_{ij} - x + y}{2} & x + y \le w_{ij} \\ (w_{ij} - x)_+ & \text{otherwise.} \end{cases} \tag{2.14}$$

It is easy to check that f is continuous everywhere in $\mathbb{R}_+^2$. Also, it is differentiable except in $\{(x,y) \in \mathbb{R}_+^2 : x+y = w_{ij} \text{ or } x = w_{ij}\}$, and satisfies $||\nabla f||_1 = |\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}| \le 1$. Hence, f is Lipschitz continuous in the L^∞ norm, with Lipschitz constant 1, i.e., it is non-expansive in sup norm.

Notice that T_κ maps $D \equiv [0,W]^{2|E|}$ into itself. The thesis follows from [14, Corollary 1].

Proof. [Proof (Theorem 1.3)] With the definitions given above, consider $W = 1$ (whence $\|\mathsf{T}\underline{\alpha}^t - \underline{\alpha}^0\|_\infty \le 1$ for all t) and apply [3, Theorem 1].

2.1 Exponentially fast convergence to unique Nash bargaining solution Convergence of the natural dynamics was studied in an earlier version of this paper using a different (and much more laborious) technique [18]. While the results in Section 1.3 constitute a large improvement in elegance and generality over those of [18], the latter retain an independent interest. Indeed the analysis of [18] shows that convergence is exponentially fast in a well defined class of instances. We decided therefore to retain the main result of that analysis (recast from [18]).

THEOREM 2.1. *Assume $W = 1$. Let G be an instance having unique Nash bargaining solution $(M, \underline{\gamma})$ with* KT *gap $\sigma > 0$, and let $\underline{\gamma}$ denote the corresponding allocation. Then, for any $\epsilon \in (0, \sigma/4)$, there exists $T_*(n,\sigma,\epsilon) = C\, n^7 \big[1/\sigma + \log(\sigma/\epsilon)\big]$,such that, for any initial condition with $\underline{\alpha}^0 \in [0,1]^{2|E|}$, and any $t \ge T_*$ the natural dynamics yields earning estimates $\underline{\gamma}^t$, with $|\gamma_i^t - \gamma_i| \le \epsilon$ for all $i \in V$. Moreover, $\underline{\alpha}^t$ induces the matching M and $(M, \underline{\gamma}^t)$ is a (4ϵ)-NB solution for any $t \ge T_*$.*

We refer to Appendix G for a definition of the KT gap σ (here KT stands for Kleinberg-Tardos). Suffice it to say that it is related to the Kleinberg-Tardos decomposition of G and that it is polynomially computable [19].

As mentioned above, the proof is based on a very different technique, namely on 'approximate decoupling' of the natural dynamics on different KT structures under the assumptions $\sigma > 0$ (which is generic) and that there is a unique NB solution. See preprint [18] for a complete proof.

Let us stress here that, for fixed σ, $T_*(n,\sigma,\epsilon)$ is only logarithmic in $(1/\epsilon)$ while it is proportional to $1/\epsilon^2$ in Theorem 1.3. In other words, for instances with KT gap bounded away from 0, the natural dynamics converges exponentially fast, while Theorem 1.3 guarantees inverse polynomial convergence in the general case.

3 Fixed point properties: Proof of Theorem 1.1

Let $\mathcal{S}$ be the set of optimum solutions of LP (1.2). We call $e \in E$ a *strong-solid edge* if $x_e^* = 1$ for all $x^* \in \mathcal{S}$ and a *non-solid edge* if $x_e^* = 0$ for all $x^* \in \mathcal{S}$. We call $e \in E$ a *weak-solid edge* if it is neither strong-solid nor non-solid.

Proof of Theorem 1.1: From fixed points to NB solutions. The direct part follows from the following set of fixed point properties. The proofs of these properties are given in Appendix D. Throughout $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ is a fixed point of the dynamics (1.4), (1.6) (with $\underline{\gamma}$ given by (1.5)).

(1) Two players $(i,j) \in E$ are called *partners* if $\gamma_i + \gamma_j = w_{ij}$. Then the following are equivalent: (a) i and j are partners, (b) $w_{ij} - \alpha_{i\backslash j} - \alpha_{j\backslash i} \ge 0$, (c) $\gamma_i = m_{j\to i}$ and $\gamma_j = m_{i\to j}$.

(2) Let $P(i)$ be the set of all partners of i. Then the following are equivalent: (a) $P(i) = \{j\}$ and $\gamma_i > 0$, (b) $P(j) = \{i\}$ and $\gamma_j > 0$, (c) $w_{ij} - \alpha_{i\backslash j} - \alpha_{j\backslash i} > 0$, (d) i and j receive unique best positive offers from each other.

(3) We say that (i,j) is a *weak-dotted edge* if $w_{ij} - \alpha_{i\backslash j} - \alpha_{j\backslash i} = 0$, *a strong-dotted edge* if $w_{ij} - \alpha_{i\backslash j} - \alpha_{j\backslash i} > 0$, and a *non-dotted edge* otherwise. If i has no adjacent dotted edges, then $\gamma_i = 0$.

(4) $\underline{\gamma}$ is an optimum solution for the dual LP (1.3) to LP (1.2) and $m_{i\to j} = (w_{ij} - \gamma_i)_+$ holds for all $(i,j) \in E$.

(5) The balance property (1.1), holds at every edge $(i,j) \in E$ (with both sides being non-negative).

(6) An edge is strong-solid (weak-solid) if and only if it is strongly (weakly) dotted.

Proof. [Proof of Theorem 1.1 (a), direct implication] Assume that the LP (1.2) has a unique optimum that is integral. Then, by property 6, the set of strong-dotted edges form the unique maximum weight matching M^* and all other edges are non-dotted. By property (3) for i that is unmatched under M^*, $\gamma_i = 0$. Hence by property (2), $\underline{\alpha}$ induces the matching M^*. Finally, by properties 4 and 5 the pair $(M^*, \underline{\gamma})$ is stable and balanced and thus forms a NB solution.

The corresponding result for the non-unique optimum case (part (b)) can be proved similarly: it follows immediately Theorem D.1, Appendix D.

REMARK 3.1. *Properties 1-6 hold for any instance. This leads to the general result Theorem D.1 in Appendix D shows that in general, fixed points correspond to dual optima satisfying the unmatched balance property (1.1).*

Proof of Theorem 1.1: From NB solutions to fixed points.

Proof. Consider any NB solution $(M, \underline{\gamma}_{\rm NB})$. Using a Proposition 1.1, M is a maximum weight matching. Construct a corresponding FP as follows. Set $m_{i\to j} = (w_{ij} - \gamma_{{\rm NB},i})_+$ for all $(i,j) \in E$. Compute $\underline{\alpha}$ using $\alpha_{i\backslash j} = \max_{k\in\partial i\backslash j} m_{k\to i}$. We claim that this is a FP and that the corresponding $\underline{\gamma}$ is $\underline{\gamma}_{\rm NB}$. To prove that we are at a fixed point, we imagine updated offers $\underline{m}^{\rm upd}$ based on $\underline{\alpha}$, and show $\underline{m}^{\rm upd} = \underline{m}$.

Consider a matching edge $(i,j) \in M$. We know that $\gamma_{{\rm NB},i} + \gamma_{{\rm NB},j} = w_{ij}$. Also stability and balance tell us $\gamma_{{\rm NB},i} - \max_{k\in\partial i\backslash j}(w_{ik} - \gamma_{{\rm NB},k})_+ = \gamma_{{\rm NB},j} - \max_{l\in\partial j\backslash i}(w_{jl} - \gamma_{{\rm NB},l})_+$ and both sides are non-negative. Hence, $\gamma_{{\rm NB},i} - \alpha_{i\backslash j} = \gamma_{{\rm NB},j} - \alpha_{j\backslash i} \geq 0$. Therefore $\alpha_{i\backslash j} + \alpha_{j\backslash i} \leq w_{ij}$,

$$m^{\rm upd}_{i\to j} = \frac{w_{ij} - \alpha_{i\backslash j} + \alpha_{j\backslash i}}{2} = \frac{w_{ij} - \gamma_{{\rm NB},i} + \gamma_{{\rm NB},j}}{2}$$
$$= \gamma_{{\rm NB},j} = w_{ij} - \gamma_{{\rm NB},i} = m_{i\to j}\,.$$

By symmetry, we also have $m^{\rm upd}_{j\to i} = \gamma_{{\rm NB},i} = m_{j\to i}$. Hence, the offers remain unchanged. Now consider $(i,j) \notin M$. We have $\gamma_{{\rm NB},i} + \gamma_{{\rm NB},j} \geq w_{ij}$ and, $\gamma_{{\rm NB},i} = \max_{k\in\partial i\backslash j}(w_{ik} - \gamma_{{\rm NB},k})_+ = \alpha_{i\backslash j}$. Similar equation holds for $\gamma_{{\rm NB},j}$. The validity of this identity can be checked individually in the cases when $i \in M$ and $i \notin M$. Hence, $\alpha_{i\backslash j} + \alpha_{j\backslash i} \geq w_{ij}$. This leads to $m^{\rm upd}_{i\to j} = (w_{ij} - \alpha_{i\backslash j})_+ = (w_{ij} - \gamma_{{\rm NB},i})_+ = m_{i\to j}$. By symmetry, we know also that $m^{\rm upd}_{j\to i} = m_{j\to i}$.

Finally, we show $\underline{\gamma} = \underline{\gamma}_{\rm NB}$. For all $(i,j) \in M$, we already found that $m_{i\to j} = \gamma_j$ and vice versa. For any edge $(ij) \notin M$, we know $m_{i\to j} = (w_{ij} - \gamma_{{\rm NB},i})_+ \leq \gamma_{{\rm NB},j}$. This immediately leads to $\underline{\gamma} = \underline{\gamma}_{\rm NB}$. It is worth noting that making use of the uniqueness of LP optimum we know that $M = M^*$, and we can further show that $\gamma_i = m_{j\to i} > \alpha_{i\backslash j}$ if and only if $(ij) \in M$, i.e., the fixed point reconstructs the pairing $M = M^*$.

4 Polynomial convergence on bipartite graphs: Proof of Theorem 1.5

Theorem 1.5 says that on a bipartite graph, under a small random perturbation on any problem instance, the natural dynamics is likely to quickly find the maximum weight matching. Now, in light of Corollary 1.1, this simply involves showing that the gap g of the perturbed problem instance is likely to be sufficiently large. We use a version of the well known Isolation Lemma to for this. Note that on bipartite graphs, there is always an integral optimum to the LP (1.2).

Next, is our Isolation lemma (recast from [15]). For the proof, see Appendix C.

Lemma 4.1. (Isolation Lemma) *Consider a bipartite graph $G = (V, E)$. Choose $\eta > 0$, $\xi > 0$. Edge weights are generated as follows: for each $e \in E$, $\bar{w}_e$ is chosen uniformly in $[w_e, w_e+\eta]$. Denote by $\mathcal{M}$ the set of matchings in G. Let M^* be a maximum weight matching. Let M^{**} be a matching having the maximum weight in $\mathcal{M}\backslash M^*$. Denote by $\bar{w}(M)$ the weight of a matching M. Then*

$$\text{(4.15)}\quad \Pr[\ \bar{w}(M^*) - \bar{w}(M^{**}) \geq \eta\xi/(2|E|)\ ] \ \geq\ 1 - \xi$$

Proof. [Proof of Theorem 1.5] Using Lemma 4.1, we know that the gap of the perturbed problem satisfies $\bar{g} \geq \eta\xi/(2|E|)$ with probability at least $1 - \xi$. Now, the weights in the perturbed instance are bounded by $\bar{W} = 2$. Rescale by dividing all weights and messages by 2, and use Corollary 1.1. The theorem follows from the following two elementary observations. First, an $(\epsilon/2)$-NB solution for the rescaled problem corresponds to an ϵ-NB solution for the original problem. Second, induced matchings are unaffected by scaling.

We remark that Theorem 1.5 does not generalize to any (non-bipartite) graph with edge weights such that the LP (1.2) has an integral optimum, for the following reason. We can easily generalize the Isolation Lemma to show that the gap g of the perturbed problem is likely to be large also in this case. However, there is a probability arbitrarily close to 1 (depending on the instance) that a random perturbation will result in an instance for which LP (1.2) does *not* have an integral optimum, i.e. the perturbed instance does not have any Nash bargaining solutions!

5 Extension: The case of unequal bargaining powers

It is reasonable to expect that not all edge surpluses on matching edges are divided equally in an exchange network setting. Some nodes are likely to have more 'bargaining power' than others. This bargaining power can arise, for example, from 'patience'; a patient agent is expected to get more than half the surplus when trading with an impatient partner. This phenomenon is well known in the Rubinstein game [25] where nodes alternately make offers to each other until an offer is accepted – the node with a smaller discount factor earns more in the subgame perfect Nash equilibrium. Moreover, a recent experimental study of bargaining in exchange networks [8] found that patience correlated positively with earnings.

A reasonable approach to model this effect would be to assign a positive 'bargaining power' to each node, and postulate that if a pair of nodes trade with each other, then the edge surplus is divided in the ratio of their bargaining powers. We choose instead, a more general setting where on each edge (ij) there is an expected surplus split fraction quantified by $r_{ij} \in (0,1)$. Namely,

r_{ij} is the fraction of surplus that goes to i if i and j trade with each other, and similarly for r_{ji}. Note that we have $r_{ij} + r_{ji} = 1$. We call a weighted graph G along with the postulated split fraction vector $\underline{r}$ an *unequal division (UD)* instance.

The balance condition is replaced by *correct division* condition

$$(5.16) \qquad [r_{ij}]^{-1}\big[\gamma_i - \max_{k\in\partial i\backslash j}(w_{ik} - \gamma_k)_+\big] \stackrel{\text{UD}}{=} [r_{ji}]^{-1}\big[\gamma_j - \max_{l\in\partial j\backslash i}(w_{jl} - \gamma_l)_+\big] \,,$$

on matched edges (ij). We retain the stability condition. We call trade outcomes satisfying (5.16) and stability *unequal division (UD) solutions*. A natural modification to our dynamics in this situation consists of the following redefinition of offers.

$$(5.17) \qquad m^t_{i\to j} \stackrel{\text{UD}}{=} (w_{ij} - \alpha^t_{i\backslash j})_+ - r_{ij}(w_{ij} - \alpha^t_{i\backslash j} - \alpha^t_{j\backslash i})_+ \,.$$

We call the dynamics resulting from (5.17) and the update rule (1.6) the *UD-natural dynamics*. One can check that T defined in (2.12) is non-expansive for offers defined as in (5.17). *It follows that Theorems 1.2 and 1.3 hold for the UD-natural dynamics with damping.* (We retain Definition 1.2 of an ϵ-FP). Further, Theorem 1.1 can also be extended to this case. The proof involves exactly the same steps as for the natural dynamics (cf. Section 3). Properties 1-6 in the direct part all hold (proofs nearly verbatim) and an identical construction works for the converse.

THEOREM 5.1. *Let G be an instance for which the LP (1.2) admits an integral optimum. Let $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ be a fixed point of the UD-natural dynamics. Then $(M^*, \underline{\gamma})$ is a UD solution for any maximum weight matching M^*. Conversely, for any UD solution $(M, \underline{\gamma}_{\text{UD}})$, matching M is a maximum weight matching and there is a unique fixed point of the UD-natural dynamics with $\underline{\gamma} = \underline{\gamma}_{\text{UD}}$.*

Further, if the LP (1.2) *has a* unique *integral optimum, corresponding to matching M^*, then any fixed point $\underline{\alpha}$ induces matching M^*.*

We note that the following generalization of the result on existence of Nash bargaining solutions [19] follows from Theorem 5.1 and the existence of fixed points.

LEMMA 5.1. *UD solutions exist if and only if a stable outcome exists (i.e. LP* (1.2) *has an integral optimum.)*

Proof. The direct part of Theorem 5.1, along with the existence of fixed points of the UD natural dynamics (from Brouwer's fixed point theorem, also first part of Theorem 1.2 for UD) shows that UD solutions exist if LP (1.2) has an integral optimum. The converse is trivial since if LP (1.2) has no integral optimum, then there are no stable solutions (see Proposition 1.1) and hence no UD solutions.

Characterizing approximate fixed points in the UD case: It is possible to derive a characterization similar to Theorem 1.4 also for the UD case. However, the bound on ϵ needed to ensure that the right pairing emerges in an ϵ-FP turns out to be exponentially small in n. As such, we are only able to show that a pairing emerges in time $2^{O(n)}/g^2$. Our work [17], shows that, in fact, it *does* take exponentially long for a pairing to emerge in worst case.

Acknowledgements. We thank Eva Tardos for introducing us to network exchange theory and Daron Acemoglu for insightful discussions. We also thank the anonymous referees for their comments.

A large part of this work was done while Y. Kanoria, M. Bayati and A. Montanari were at Microsoft Research New England. This research was partially supported by NSF, grants CCF-0743978 and CCF-0915145, and by a Terman fellowship. Y. Kanoria is supported by a 3Com Corporation Stanford Graduate Fellowship.

References

[1] B. Aspvall and Y. Shiloach, *A polynomial time algorithm for solving systems of linear inequalities with two variables pre inequality*, Proc. 20th IEEE Symp. Found. of Comp. Sc., 1979.

[2] Y. Azar, B. Birnbaum, L. Elisa Celis, N. R. Devanur and Y. Peres, *Convergence of Local Dynamics to Balanced Outcomes in Exchange Networks*, Proc. 50th IEEE Symp. Found. of Comp. Sc., 2009.

[3] J. Baillon and R. E. Bruck, *The rate of asymptotic regularity is $O(1/\sqrt{n})$*, in: A.G. Kartsatos (ed.), Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math. 178, Marcel Dekker, Inc., New York, 1996, pp. 51–81.

[4] M. Bateni, M. Hajiaghayi, N. Immorlica and H. Mahini, *The cooperative game theory foundations of network bargaining games*, Proc. Intl. Colloq. Automata, Languages and Programming, 2010.

[5] M. Bayati, D. Shah and M. Sharma, *Max-Product for Maximum Weight Matching: Convergence, Correctness, and LP Duality*, IEEE Trans. Inform. Theory, 54 (2008), pp. 1241–1251.

[6] M. Bayati, C. Borgs, J. Chayes, R. Zecchina, *On the exactness of the cavity method for Weighted b-Matchings on Arbitrary Graphs and its Relation to Linear Programs*, arXiv:0807.3159, (2007).

[7] D. P. Bertsekas, *The Auction Algorithm: A Distributed Relaxation Method for the Assignment Problem*, Annals of Operations Research, 14 (1988), pp. 105–123.

[8] T. Chakraborty, S. Judd, M. Kearns, J. Tan, *A Behavioral Study of Bargaining in Social Networks*, Proc. 10th ACM Conf. Electronic Commerce, 2010.
[9] T. Chakraborty, M. Kearns and S. Khanna, *Network Bargaining: Algorithms and Structural Results*, Proc. 10th ACM Conf. Electronic Commerce, July 2009.
[10] K. S. Cook and T. Yamagishi, *Power exchange in networks: A power-dependence formulation*, Social Networks, 14 (1992), pp. 245–265.
[11] C. Daskalakis, C. Papadimitriou, *On oblivious PTAS's for nash equilibrium*, Proc. ACM Symp. Theory of Computing, 2009.
[12] M. Edelstein and R.C. O'Brien, *Nonexpansive mappings, asymptotic regularity, and successive approximations*, J. London Math. Soc. 1 (1978), pp. 547–554.
[13] U. Faigle, W. Kern, and J. Kuipers, *On the computation of the nucleolus of a cooperative game*, International Journal of Game Theory, 30 (2001), pp. 79–98.
[14] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. American Mathematical Society, 59(1) (1976).
[15] D. Gamarnik, D. Shah and Y. Wei, *Belief Propagation for Min-cost Network Flow: Convergence and Correctness*, Proc. ACM-SIAM Symp. Disc. Alg., 2010.
[16] B. Huang, T. Jebara, *Loopy belief propagation for bipartite maximum weight b-matching*, Artificial Intelligence and Statistics (AISTATS), March, 2007.
[17] Y. Kanoria, *An FPTAS for Bargaining Networks with Unequal Bargaining Powers*, Proc. Workshop on Internet and Network Economics, 2010, (full version arXiv:1008.0212).
[18] Y. Kanoria, M. Bayati, C. Borgs, J. Chayes, and A. Montanari, *A Natural Dynamics for Bargaining on Exchange Networks*, arXiv:0912.5176 (2009).
[19] J. Kleinberg and E. Tardos, *Balanced outcomes in social exchange networks*, in Proc. ACM Symp. Theory of Computing, 2008.
[20] U. Kohlenback, *A Quantitative Version Of A Theorem Due To Borwein-Reich-Shafrir*, Numerical Functional Analysis and Optimization, 22(5-6), August 2001.
[21] J.W. Lucas, C.W. Younts, M.J. Lovaglia, and B. Markovsky, *Lines of power in exchange networks*, Social Forces, 80 (2001), pp. 185–214.
[22] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math Soc., 4 (1953), pp. 506–510.
[23] J. Nash, *The bargaining problem*, Econometrica, 18 (1950), pp. 155–162.
[24] S.C. Rochford, *Symmetric pairwise-bargained allocations in an assignment market*, in J. Economic Theory, 34 (1984), pp. 262–281.
[25] A. Rubinstein: *Perfect equilibrium in a bargaining model*, Econometrica, 50 (1982), pp. 97–109.
[26] S. Sanghavi, D. Malioutov, A. Willsky, *Linear Programming Analysis of Loopy Belief Propagation for Weighted Matching*, Proc. Neural Inform. Processing Systems, 2007.
[27] A. Schrijver, *Combinatorial Optimization*, Springer-Verlag, Vol. A (2003).
[28] M. Sotomayor, *On The Core Of The One-Sided Assignment Game*, 2005, `http://www.usp.br/feaecon/media/fck/File/one_sided_assignment_game.pdf`.
[29] J. Skvoretz and D. Willer, *Exclusion and power: A test of four theories of power in exchange networks*, American Sociological Review, 58 (1993), pp. 801–818.
[30] F. Spitzer, *Principles of Random Walk*, Springer, 2005.
[31] M. J. Wainwright, T. S. Jaakkola, and A. S. Willsky, *Exact MAP estimates via agreement on (hyper)trees: Linear programming and message-passing*, IEEE Trans. Inform. Theory, 51 (2005), pp. 3697–3717.
[32] D. Willer (ed.) *Network Exchange Theory*, Praeger, 1999.

A Polynomial time local algorithm for balancing given max weight matching

In this section, we show the following:

THEOREM A.1. *Consider any instance admitting a NB solution, i.e. such that the LP* (1.2) *has an integral optimum. There is a local algorithm that takes a maximum weight matching M^* as input, and constructs an ϵ-NB solution with computational effort $O(|E|^2 + |E|/\epsilon^2) = O(\text{poly}(|V|, 1/\epsilon))$.*

Note that we require no conditions on the LP (1.2), other than that it possesses an integral optimum. The LP gap g may be arbitrarily small, or even 0. Thus, if a polynomial time local algorithm for finding maximum weight matching is discovered, Theorem A.1 directly implies a polynomial time local algorithm for finding ϵ-NB solutions.

Our local procedure involves two steps:

(1) Construct a dual optimum, i.e., a stable allocation. This takes at most $2|E|$ iterations of message passing as described in Section A.1.

(2) Run edge balancing that preserves stability and leads to ϵ-balance as in our work [17].

A.1 Constructing a dual optimum from M^*

Our definition of ϵ-NB solutions retains a strict version of stability while relaxing the balance requirement to ϵ balance (cf. 1.2). In the first phase of our local algorithm, we use max-product belief propagation to find a stable allocation, *given a maximum weight matching M^**. This is achieved locally and in polynomial time.

We use $\underline{\alpha}$ and $\underline{m}$ for max-product BP messages in the rest of this subsection. (cf. Appendix F). Consider the standard *undamped synchronous BP updates* given by:

$$m^t_{i\to j} = (w_{ij} - \alpha^t_{i\backslash j})_+$$
$$\alpha^{t+1}_{i\backslash j} = \max_{k\in\partial i\backslash j} m^t_{k\to i} \tag{A.1}$$

We use a carefully chosen initialization (different from the usual all-zero) to achieve our objective:

$$m^0_{i\to j} = \begin{cases} w_{ij} & \text{if } (ij) \in M^* \\ 0 & \text{otherwise} \end{cases} \tag{A.2}$$

Let the version of max-product BP message passing defined by (A.1) and (A.2) be denoted by $\mathcal{A}$.

Our key result on $\mathcal{A}$ is the following:

CLAIM 1. *Algorithm $\mathcal{A}$ converges to an exact fixed point in $2|E|$ iterations.*

A stable allocation follows immediately from a fixed point $\underline{m}^*$ of $\mathcal{A}$ (see Section F, Eq. (F.14)). The allocation γ_i of node i is simply the mean of the two largest values in the set $\{m_{k\to i} : k \in \partial i\}$ (if i has only one neighbor, take the second largest offer to be 0).

We omit the proof of Claim 1 in the interest of space (the full version of our related work [17] contains a proof).

A.2 Balancing a stable allocation The balancing phase proceeds via the algorithm EDGE REBALANCING in our work [17], specialized to the balanced case. The key idea is that rebalancing updates preserve stability, while ensuring that balance is quickly achieved.

B Variations of the natural dynamics

Now that we have a reasonable dynamics that converges fast to balanced outcomes, it is natural ask the question "What can be say about variations of the natural dynamics that would also be expected to yield balanced outcomes?" Can we handle asynchronous updates, different nodes updating at different rates, damping factors that vary across nodes and in time, and so on? We discuss some of these questions in this section, focussing on some situations in which we *can* prove convergence with minimal additional work. Note that we are only concerned with extending our convergence results since the fixed point properties remain unchanged.

B.1 Node dependent damping Consider that the damping factor may be different for different nodes, but unchanging over time. Denote by $\kappa(v)$, the damping factor for node v. Assume that $\kappa(v) \in [1-\kappa^*, \kappa^*]$ $\forall v \in V$ for some $\kappa^* \in [0.5, 1)$, i.e. damping factors are uniformly bounded away from 0 and 1. Define operator $\mathsf{T} : [0,1]^{|E|} \to [0,1]^{|E|}$ by

$$(\mathsf{T}\underline{\alpha})_{i\backslash j} = \left(\frac{\kappa(i)}{\kappa^*}\right) \max_{k\in\partial i\backslash j} m_{k\to i} + \left(\frac{\kappa^* - \kappa(i)}{\kappa^*}\right) \alpha_{i\backslash j} \tag{B.3}$$

T is non-expansive. Now, the dynamics can be written as $\underline{\alpha}^{t+1} = \kappa^*\mathsf{T}\underline{\alpha}^t + (1-\kappa^*)\underline{\alpha}^t$. Clearly, convergence to fixed points (Theorem 1.2) holds in this situation. Note that fixed points of T are the same as fixed points of the natural dynamics. Moreover, we can use [3] to assert that $||\underline{\alpha}^t - \mathsf{T}\underline{\alpha}^t||_\infty = O(1/\sqrt{t})$ and hence $\underline{\alpha}^t$ is an $O(1/\sqrt{t})$-FP. In short, we don't lose anything with this generalization!

B.2 Time varying damping Now consider instead that the damping may change over time, but is the same for all nodes. Denote by κ_t the damping factor at time t, i.e.

$$\underline{\alpha}^{t+1} = \kappa_t \max_{k\in\partial i\backslash j} m^t_{k\to i} + (1-\kappa_t)\underline{\alpha}^t \tag{B.4}$$

The result of [14] implies that, as long as $\sum_{t=0}^\infty \kappa_t = \infty$ and $\lim_{t\to\infty}\sup\kappa_t < 1$, the dynamics is guaranteed to converge to a fixed point. Note that again the fixed points are unchanged. [20] provides a quantitative estimate of the rate of convergence in this case, guaranteeing in particular that an ϵ-FP is reached in time $\exp(O(1/\epsilon))$ if κ_t is uniformly bounded away from 0 and 1. Note that this estimate is much weaker than the one provided by [3], leading to Theorem 1.3. It seems intuitive that the stronger $O(1/\sqrt{k})$ bound holds also for the time varying damping case in the general non-expansive operator setting, but a proof has remained elusive thus far.

B.3 Asynchronous updates Finally, we look at the case of asynchronous updates, i.e., one message $\alpha_{i\backslash j}$ is updated in any given step while the others remain unchanged. Define $\mathsf{T}_{i\backslash j} : [0,1]^{|E|} \to [0,1]^{|E|}$ by

$$(\mathsf{T}_{i\backslash j}\underline{\alpha})_{i'\backslash j'} = \begin{cases} \max_{k\in\partial i'\backslash j'} m_{k\to i'} & \text{if } (i,j) = (i',j') \\ \alpha_{i'\backslash j'} & \text{otherwise} \end{cases} \tag{B.5}$$

Let $m \equiv |E|$. There are $2m$ such operators, two for each edge. Clearly, each $\mathsf{T}_{i\backslash j}$ is non-expansive in sup-norm. Now, consider an arbitrary permutation of the $2m$ directed edges $((i_1,j_1),(i_2,j_2),\ldots)$. Consider the updates induced by $\mathsf{T}_{i_1\backslash j_1}, \mathsf{T}_{i_2\backslash j_2}, \ldots$ in order, each with a damping factor of $1/(2m)$. Consider the resulting product

$$\begin{aligned} &\Big((1/2m)\mathsf{T}_{i_1\backslash j_1} + (1-(1/2m))\,I\Big)\cdot \\ &\cdot\Big((1/2m)\mathsf{T}_{i_2\backslash j_2} + (1-(1/2m))\,I\Big)\ \cdot\ \ldots \\ &\qquad = \Big(1-(1-(1/2m))^{2m}\Big)\mathsf{T} + (1-(1/2m))^{2m}\, I \end{aligned} \tag{B.6}$$

Here (B.6) defines T, and I is the identity operator. It is easy to deduce that T above is non-expansive from the following elementary facts – the product of non-expansive operators in non-expansive, and the convex combination of non-expansive operators is non-expansive. Also, $(1-(1/2m))^{2m} \in [1/4, 1/e]\ \forall m$. Thus, if we repeat these asynchronous updates periodically in a series of 'update cycles', we are guaranteed to quickly converge to an ϵ-FP of T ([3]).

PROPOSITION B.1. *An ϵ-FP of T is an $O(m\epsilon)$-FP of the natural dynamics.*

Proof. Suppose we start an update cycle at $\underline{\alpha}$, an ϵ-FP of T. Then we know that at the end of the update cycle, no coordinate changes by more than $(1-1/4)\epsilon \le \epsilon$. Note that among the $2m$ steps in a cycle, any particular $i\backslash j$ 'coordinate' only changes in one step. Thus, each such coordinate change is bounded by ϵ. Consider the s-th step in the update cycle. The state before the s-th step, call it $\underline{\alpha}(s-1)$, is ϵ-close to $\underline{\alpha}$. Also, we know that the $(i_s\backslash j_s)$ coordinate changes by at most ϵ in this step. Hence,

$$\begin{aligned} & ||\mathsf{T}_{i_s\backslash j_s}\underline{\alpha}(s-1) - \underline{\alpha}(s-1)||_\infty \le (2m)\epsilon \\ \Rightarrow \quad & ||\mathsf{T}_{i_s\backslash j_s}\underline{\alpha} - \underline{\alpha}||_\infty \le (2m+2)\epsilon \end{aligned}$$

This holds for $s = 1, 2, \ldots, 2m$. Hence the result.

Note that with $\epsilon = 0$, Proposition B.1 tells us that fixed points of T are fixed points of the natural dynamics. Thus, we are immediately guaranteed convergence to fixed points of the natural dynamics. Moreover, the quantitative estimate in Proposition B.1 guarantees that in a small number of update cycles we reach approximate fixed points of the natural dynamics.

Finally, we comment that instead of ordering updates by a permutation of directed edges, we could have an arbitrary periodic sequence of updates satisfying non-starvation and obtain similar results. For example, this would include cases where some nodes update more frequently than others. Also, note that the damping factors of $(1/2m)$ were chosen for simplicity and to ensure *fast* convergence. Any non-trivial damping would suffice to guarantee convergence.

It remains an open question to show convergence for non-periodic asynchronous updates.

C Proof of Isolation lemma

Our proof of the isolation lemma is adapted from [15].

Proof. [Proof of Lemma 4.1] Fix $e \in E$ and fix $\bar{w}_{e'}$ for all $e' \in E\backslash e$. Let M_e be a maximum weight matching among matchings that strictly include edge e, and let $M_{\sim e}$ be a maximum weight matching among matchings that exclude edge e. Clearly, M_e and $M_{\sim e}$ are independent of $\bar{w}_e$. Define

$$\begin{aligned} f_e(\bar{w}_e) \equiv \bar{w}(M_e) &= f_e(0) + \bar{w}_e \\ f_{\sim e} \equiv \bar{w}(M_{\sim e}) &= \text{const} < \infty \end{aligned}$$

Clearly, $f_e(0) \le f_{\sim e}$, since we cannot do worse by forcing exclusion of a zero weight edge. Thus, there is some unique $\theta \ge 0$ such that $f_e(\theta) = f_{\sim e}$. Define $\delta = \eta\xi/2|E|$. Let $D(e)$ be the event that $|\bar{w}(M_e) - \bar{w}(M_{\sim e})| < \delta$. It is easy to see that $D(e)$ occurs iff $\bar{w}_e \in (\theta-\delta, \theta+\delta)$. Thus, $\Pr[D(e)] \le 2\delta/\eta = \xi/|E|$. Now,

$$\text{(C.7)} \qquad \left\{\bar{w}(M^*) - \bar{w}(M^{**}) < \delta\right\} = \bigcup_{e\in E} D(e)$$

and the lemma follows by union bound.

D Proofs of fixed point properties

In this section we state and prove the fixed point properties that were used for the proof of Theorem 1.1 in Section 3. Before that, however, we remark that the condition: "LP (1.2) has a unique optimum" in Theorem 1.1(a) is almost always valid.

REMARK D.1. *We argue that the condition "LP (1.2) has a unique optimum" is generic in instances with integral optimum:*
Let $\mathsf{G}_{\rm I} \subset [0, W]^{|E|}$ be the set of instances having an integral optimum. Let $\mathsf{G}_{\rm UI} \subset \mathsf{G}_{\rm I}$ be the set of instances having a unique integral optimum. It turns out that $\mathsf{G}_{\rm I}$ has dimension $|E|$ (i.e. the class of instances having an integral optimum is large) and that $\mathsf{G}_{\rm UI}$ is both open and dense in $\mathsf{G}_{\rm I}$.

Notation. In proofs of this section and Section E we denote surplus $w_{ij} - \alpha_{i\backslash j} - \alpha_{j\backslash i}$ of edge (ij) by $\mathcal{S}\text{urp}_{ij}$.

LEMMA D.1. *$\underline{\gamma}$ satisfies the constraints of the dual problem (1.3).*

Proof. Since offers $m_{i\to j}$ are by definition non-negative therefore for all $v \in V$ we have $\gamma_v \ge 0$. So we only need to show $\gamma_i + \gamma_j \ge w_{ij}$ for any edge $(ij) \in E$. It is easy to see that $\gamma_i \ge \alpha_{i\backslash j}$ and $\gamma_j \ge \alpha_{i\backslash j}$. Therefore, if $\alpha_{i\backslash j} + \alpha_{i\backslash j} \ge w_{ij}$ then $\gamma_i + \gamma_j \ge w_{ij}$ holds and we are done. Otherwise, for $\alpha_{i\backslash j} + \alpha_{i\backslash j} < w_{ij}$ we have $m_{i\to j} = \frac{w_{ij} - \alpha_{i\backslash j} + \alpha_{j\backslash i}}{2}$ and $m_{j\to i} = \frac{w_{ij} - \alpha_{j\backslash i} + \alpha_{i\backslash j}}{2}$ which gives $\gamma_i + \gamma_j \ge m_{i\to j} + m_{j\to i} = w_{ij}$.

Recall that for any $(ij) \in E$, we say that i and j are 'partners' if $\gamma_i + \gamma_j = w_{ij}$ and $P(i)$ denotes the partners of node i. In other words $P(i) = \{j : j \in \partial i, \gamma_i + \gamma_j = w_{ij}\}$.

Lemma D.2. *The following are equivalent:*
(a) i and j are partners,
(b) $\mathcal{S}urp_{ij} \geq 0$,
(c) $\gamma_i = m_{j\to i}$ and $\gamma_j = m_{i\to j}$.
Moreover, if $\gamma_i = m_{j\to i}$ and $\gamma_j > m_{i\to j}$ then $\gamma_i = 0$.

Proof. We will prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$: Since $\gamma_i \geq \alpha_{i\setminus j}$ and $\gamma_j \geq \alpha_{j\setminus i}$ always holds then $w_{ij} = \gamma_i + \gamma_j \geq \alpha_{i\setminus j} + \alpha_{j\setminus i}$.

$(b) \Rightarrow (c)$: If $\mathcal{S}\mathrm{urp}_{ij} \geq 0$ then $(w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2 \geq \alpha_{j\setminus i}$. But $m_{i\to j} = (w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2$ therefore $\gamma_j = m_{i\to j}$. The argument for $\gamma_i = m_{j\to i}$ is similar.

$(c) \Rightarrow (a)$: If $\mathcal{S}\mathrm{urp}_{ij} \geq 0$ then $m_{i\to j} = (w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2$ and $m_{j\to i} = (w_{ij} - \alpha_{j\setminus i} + \alpha_{i\setminus j})/2$ which gives $\gamma_i + \gamma_j = m_{i\to j} + m_{j\to i} = w_{ij}$ and we are done. Otherwise, we have $\gamma_i + \gamma_j = m_{i\to j} + m_{j\to i} \leq (w_{ij} - \alpha_{i\setminus j})_+ + (w_{ij} - \alpha_{j\setminus i})_+ < \max\Big[(w_{ij} - \alpha_{i\setminus j})_+, (w_{ij} - \alpha_{j\setminus i})_+, 2w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i}\Big] \leq w_{ij}$ which contradicts Lemma D.1 that $\underline{\gamma}$ satisfies the constraints of the dual problem (1.3).

Finally, we need to show that $\gamma_i = m_{j\to i}$ and $\gamma_j > m_{i\to j}$ give $\gamma_i = 0$. First note that by equivalence of (b) and (c) we should have $w_{ij} < \alpha_{i\setminus j} + \alpha_{j\setminus i}$. On the other hand $\alpha_{i\setminus j} \leq \gamma_i = m_{j\to i} \leq (w_{ij} - \alpha_{j\setminus i})_+$. Now if $w_{ij} - \alpha_{j\setminus i} > 0$ we get $\alpha_{i\setminus j} \leq w_{ij} - \alpha_{j\setminus i}$ which is a contradiction. Therefore $\gamma_i = (w_{ij} - \alpha_{j\setminus i})_+ = 0$.

Lemma D.3. *The following are equivalent:*
(a) $P(i) = \{j\}$ and $\gamma_i > 0$,
(b) $P(j) = \{i\}$ and $\gamma_j > 0$,
(c) $w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} > 0$.
(d) i and j receive unique best positive offers from each other.

Proof. $(a) \Rightarrow (c) \Rightarrow (b)$: (a) means that for all $k \in \partial i\setminus j$, $\mathcal{S}\mathrm{urp}_{ik} < 0$. This means $m_{k\to i} = (w_{ik} - \alpha_{k\setminus i})_+ < \alpha_{i\setminus k} = m_{j\to i}$ (using $\gamma_i > 0$). Hence, $\alpha_{i\setminus j} < m_{j\to i}$. From (a), it also follows that $m_{j\to i} > 0$ or $(w_{ij} - \alpha_{j\setminus i})_+ = w_{ij} - \alpha_{j\setminus i}$. Therefore, $m_{j\to i} \leq (w_{ij} - \alpha_{j\setminus i})_+ = w_{ij} - \alpha_{j\setminus i}$ which gives $w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} > 0$ or (c). From this we can explicitly write $m_{i\to j} = (w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2$ which is strictly bigger than $\alpha_{j\setminus i}$. Hence we obtain (b).

By symmetry $(b) \Rightarrow (c) \Rightarrow (a)$. Thus, we have shown that (a), (b) and (c) are equivalent.

$(c) \Rightarrow (d)$: (c) implies that $m_{i\to j} = (w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2 > \alpha_{j\setminus i} = \max_{k\in\partial j\setminus i} m_{k\to j}$. Thus, j receives its unique best positive offer from i. Using symmetry, it follows that (d) holds.

$(d) \Rightarrow (c)$: (d) implies $\gamma_i = m_{j\to i}$ and $\gamma_j = m_{i\to j}$. By Lemma D.2, i and j are partners, i.e. $\gamma_i + \gamma_j = w_{ij}$. Hence, $m_{i\to j} + m_{j\to i} = w_{ij}$. But since (d) holds, $\alpha_{i\setminus j} < m_{j\to i}$ and $\alpha_{j\setminus i} < m_{i\to j}$. This leads to (c).

This finishes the proof.

Recall that (ij) is a weak-dotted edge if $w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} = 0$, a strong-dotted edge if $w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} > 0$, and a non-dotted edge otherwise. Basically, for any dotted edge (ij) we have $j \in P(i)$ and $i \in P(j)$.

Corollary D.1. *One corollary of Lemmas D.2-D.3 is that strong-dotted edges are only adjacent to non-dotted edges. Also each weak-dotted edge is adjacent to at least one weak-dotted edge at each end (in both cases, assume that the earning of the two endpoints are non-zero.*

Lemma D.4. *If i has no adjacent dotted edges, then $\gamma_i = 0$*

Proof. Assume that the largest offer to i comes from j. Therefore, $\alpha_{i\setminus j} \leq m_{j\to i} \leq (w_{ij} - \alpha_{j\setminus i})_+$. Now if $w_{ij} - \alpha_{j\setminus i} > 0$ then $\alpha_{i\setminus j} \leq w_{ij} - \alpha_{j\setminus i}$ or (ij) is dotted edge which is impossible. Thus, $w_{ij} - \alpha_{j\setminus i} = 0$ and $\gamma_i = 0$.

Lemma D.5. *The following are equivalent:*
(a) $\alpha_{i\setminus j} = \gamma_i$,
(b) $\mathcal{S}urp_{ij} \leq 0$,
(c) $m_{i\to j} = (w_{ij} - \alpha_{i\setminus j})_+$.

Proof. $(a) \Rightarrow (b)$: Follows from Lemma D.3, since $\alpha_{i\setminus j} = \gamma_i$ gives $|P(i)| > 1$.

$(b) \Rightarrow (c)$: Follows from the definition of $m_{i\to j}$.

$(c) \Rightarrow (a)$: From $m_{i\to j} = (w_{ij} - \alpha_{i\setminus j})_+$ we have $\mathcal{S}\mathrm{urp}_{ij} \leq 0$. Therefore, $m_{j\to i} = (w_{ij} - \alpha_{j\setminus i})_+ \leq \max\big[w_{ij} - \alpha_{j\setminus i}, 0\big] \leq \alpha_{i\setminus j}$.

Note that (c) is symmetric in i and j, so (a) and (b) can be transformed by interchanging i and j.

Corollary D.2. $\alpha_{i\setminus j} = \gamma_i$ *iff* $\alpha_{j\setminus i} = \gamma_j$

Lemma D.6. $m_{i\to j} = (w_{ij} - \gamma_i)_+$ *holds* $\forall\, (ij) \in E$

Proof. If $w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} \leq 0$ then the result follows from Lemma D.5. Otherwise, (ij) is strongly dotted and $\gamma_i = m_{j\to i} = (w_{ij} - \alpha_{j\setminus i} + \alpha_{i\setminus j})/2$, $\gamma_j = m_{i\to j} = (w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2$. From here we can explicitly calculate $w_{ij} - \gamma_i = (w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2 = m_{i\to j}$.

Lemma D.7. *The unmatched balance property, equation* (1.1), *holds at every edge $(ij) \in E$, and both sides of the equation are non-negative.*

Proof. In light of lemma D.6, (1.1) can be rewritten at a fixed point as

$$\gamma_i - \alpha_{i\setminus j} = \gamma_j - \alpha_{j\setminus i} \qquad \text{(D.8)}$$

which is easy to verify. The case $\mathcal{S}\mathrm{urp}_{ij} \le 0$ leads to both sides of Eq. (D.8) being 0 by Corollary D.2. The other case $\mathcal{S}\mathrm{urp}_{ij} > 0$ leads to

$$m_{i\to j} - \alpha_{j\setminus i} = m_{j\to i} - \alpha_{i\setminus j} = \frac{\mathcal{S}\mathrm{urp}_{ij}}{2} \tag{D.9}$$

Clearly, we have $\gamma_i = m_{j\to i}$ and $\gamma_j = m_{i\to j}$. So Eq. (D.8) holds.

Next lemmas show that dotted edges are in correspondence with the solid edges that were defined in Section 3.

Lemma D.8. *A non-solid edge cannot be a dotted edge, weak or strong.*

Before proving the lemma let us define alternating paths. A path $P = (i_1, i_2, \ldots, i_k)$ in G is called *alternating path* if: (a) There exist a partition of edges of P into two sets A, B such that either $A \subset M^*$ or $B \subset M^*$. Moreover A (B) consists of all *odd (even)* edges; i.e. $A = \{(i_1, i_2), (i_3, i_4), \ldots\}$ $(B = \{(i_2, i_3), (i_4, i_5), \ldots\})$. (b) The path P might intersect itself or even repeat its own edges but no edge is repeated immediately. That is, for any $1 \le r \le k-2$: $\quad i_r \neq i_{r+1}$ and $i_r \neq i_{r+2}$. P is called an *alternating cycle* if $i_1 = i_k$.

Also, consider $\underline{x}^*$ and $\underline{y}^*$ that are optimum solutions for the LP and its dual, (1.2) and (1.3). The complementary slackness conditions (see [27]) for more details) state that for all $v \in V$, $y_v^*(\sum_{e\in\partial v} x_e^* - 1) = 0$ and for all $e = (ij) \in E$, $x_e^*(y_i^* + y_j^* - w_{ij}) = 0$. Therefore, for all solid edges the equality $y_i^* + y_j^* = w_{ij}$ holds. Moreover, any node $v \in V$ is adjacent to a solid edge iff $y_v^* > 0$.

Proof. [Proof of Lemma D.8] First, we refine the notion of solid edges by calling an edge e, 1-$\underline{x}^*$-solid ($\frac{1}{2}$-$\underline{x}^*$-solid) whenever $x_e^* = 1$ ($x_e^* = \frac{1}{2}$).

We need to consider two cases:

Case (I). Assume that LP has an optimum solution $\underline{x}^*$ that is integral as well (having a tight LP).

The idea of the proof is that if there exists a non-solid edge e which is dotted, we use a similar analysis to [6] to construct an alternating path consisting of dotted and $\underline{x}^*$-solid edges that leads to creation of at an optimal solution to LP (1.2) that assigns a positive value to e. This contradicts the non-solid assumption on e.

Now assume the contrary: take (i_1, i_2) that is a non-solid edge but it is dotted. Consider an endpoint of (i_1, i_2). For example take i_2. Either there is a $\underline{x}^*$-solid edge attached to i_2 or not. If there is not, we stop. Otherwise, assume (i_2, i_3) is a $\underline{x}^*$-solid edge. Using Lemma D.4, either $\gamma_{i_3} = 0$ or there is a dotted edge connected to i_3. But if this dotted edge is (i_2, i_3) then $P(i_2) \supseteq \{i_1, i_3\}$. Therefore, by Lemma D.3 there has to be another dotted edge (i_3, i_4) connected to i_3. Now, depending on whether i_4 has (has not) an adjacent $\underline{x}^*$-solid edge we continue (stop) the construction. A similar procedure could be done by starting at i_1 instead of i_2. Therefore, we obtain an alternating path $P = (i_{-k}, \ldots, i_{-1}, i_0, i_1, i_2, \ldots, i_\ell)$ with all odd edges being dotted and all even edges being $\underline{x}^*$-solid. Using the same argument as in [6] one can show that one of the following four scenarios occur.

Path: Before P intersects itself, both end-points of the path stop. Either the last edge is $\underline{x}^*$-solid (then $\gamma_v = 0$ for the last node) or the last edge is a dotted edge. Now consider a new solution $\underline{x}'$ to LP (1.2) by $x'_e = x^*_e$ if $e \notin P$ and $x'_e = 1 - x^*_e$ if $e \in P$. It is easy to see that $\underline{x}'$ is a feasible LP solution at all points $v \notin P$ and also for internal vertices of P. The only nontrivial case is when $v = i_{-k}$ (or $v = i_\ell$) and the edge (i_{-k}, i_{-k+1}) (or $(i_{\ell-1}, i_\ell)$) is dotted. In both of these cases, by construction v is not connected to an $\underline{x}^*$-solid edge outside of P. Hence, making any change inside of P is safe. Now denote the weight of all solid (dotted) edges of P by $w(P_{\mathrm{solid}})$ ($w(P_{\mathrm{dotted}})$). Here, we only include edges outside P_{solid} in P_{dotted}. Clearly,

$$\sum_{e\in E} w_e x^*_e - \sum_{e\in E} w_e x'_e = w(P_{\mathrm{solid}}) - w(P_{\mathrm{dotted}}). \tag{D.10}$$

But $w(P_{\mathrm{dotted}}) = \sum_{v\in P} \gamma_v$. Moreover, from Lemma D.1, $\underline{\gamma}$ is dual feasible which gives $w(P_{\mathrm{solid}}) \le \sum_{v\in P} \gamma_v$. We are using the fact that if there is a $\underline{x}^*$-solid edge at an endpoint of P the γ of the endpoint should be 0. Now Eq. (D.10) reduces to $\sum_{e\in E} w_e x^*_e - \sum_{e\in E} w_e x'_e \le 0$. This contradicts that $e = (i_1, i_2)$ is non-solid since $x'_e > 0$.

Cycle: P intersects itself and will contain an even cycle C_{2s}. This case can be handled very similar to the path by defining $x'_e = x^*_e$ if $e \notin C_{2s}$ and $x'_e = 1 - x^*_e$ if $e \in C_{2s}$. The proof is even simpler since the extra check for the boundary condition is not necessary.

Blossom: P intersects itself and will contain an odd cycle C_{2s+1} with a path (stem) P' attached to the cycle at point u. In this case let $x'_e = x^*_e$ if $e \notin P' \cup C_{2s+1}$, and $x'_e = 1 - x^*_e$ if $e \in P'$, and $x'_e = \frac{1}{2}$ if $e \in C_{2s+1}$. From here, we drop the subindex $2s+1$ to simplify the notation. Since the cycle has odd length, both neighbors of u in C have to be dotted. Therefore,

$$\begin{aligned}
&\sum_{e\in E} w_e x^*_e - \sum_{e\in E} w_e x'_e \\
&= w(P'_{\mathrm{solid}}) + w(C_{\mathrm{solid}}) - w(P'_{\mathrm{dotted}}) \\
&\quad - \frac{w(C_{\mathrm{dotted}}) + w(C_{\mathrm{solid}})}{2} \\
&= w(P'_{\mathrm{solid}}) + \frac{w(C_{\mathrm{solid}})}{2} - w(P'_{\mathrm{dotted}}) - \frac{w(C_{\mathrm{dotted}})}{2} .
\end{aligned}$$

Plugging $w(P'_{\text{solid}}) \le \sum_{v\in P'} \gamma_v$, $w(C_{\text{solid}}) \le \sum_{v\in C} \gamma_v - \gamma_u$, $w(P'_{\text{dotted}}) = \sum_{v\in P'} \gamma_v - \gamma_u$ and $w(C_{\text{dotted}}) = \sum_{v\in C} \gamma_v + \gamma_u$, we obtain

$$\sum_{e\in E} w_e x_e^* - \sum_{e\in E} w_e x'_e \le 0\,,$$

which is again a contradiction.

Bicycle: P intersects itself at least twice and will contain two odd cycles C_{2s+1} and $C'_{2s'+1}$ with a path (stem) P' that is connecting them. Very similar to Blossom, let $x'_e = x_e^*$ if $e \notin P' \cup C \cup C'$, $x'_e = 1 - x_e^*$ if $e \in P'$, and $x'_e = \frac{1}{2}$ if $e \in C \cup C'$. The proof follows similar to the case of blossom.

Case (II). Assume that there is an optimum solution $\underline{x}^*$ of LP that is not necessarily integral.

Everything is similar to Case (I) but the algebraic treatments are slightly different. Some edges e in P can be $\frac{1}{2}$-$\underline{x}^*$-solid ($x_e^* = \frac{1}{2}$). In particular some of the odd edges (dotted edges) of P can now be $\frac{1}{2}$-$\underline{x}^*$-solid. But the subset of $\frac{1}{2}$-$\underline{x}^*$-solid edges of P can be only sub-paths of odd length in P. On each such sub-path defining $\underline{x}' = 1 - \underline{x}^*$ means we are not affecting $\underline{x}^*$. Therefore, all of the algebraic calculations should be considered on those sub-paths of P that have no $\frac{1}{2}$-$\underline{x}^*$-solid edge which means both of their boundary edges are dotted.

Path: Define $\underline{x}'$ as in Case (I). Using the discussion above, let $P^{(1)}, \ldots, P_{(r)}$ be disjoint sub-paths of P that have no $\frac{1}{2}$-$\underline{x}^*$-solid edge. Thus, $\sum_{e\in E} w_e x_e^* - \sum_{e\in E} w_e x'_e = \sum_{i=1}^{r} \left[w(P^{(i)}_{\text{solid}}) - w(P^{(i)}_{\text{dotted}})\right]$. Since in each $P^{(i)}$ the two boundary edges are dotted, $w(P^{(i)}_{\text{solid}}) \le \sum_{v\in P^{(i)}} \gamma_v$ and $\sum_{v\in P^{(i)}} \gamma_v = w(P^{(i)}_{\text{dotted}})$. The rest can be done as in Case (I).

Cycle, Blossom, Bicycle: These cases can be done using the same method of breaking the path and cycles into sub-paths $P^{(i)}$ and following the case of path.

Lemma D.9. *Every strong-solid edge is a strong-dotted edge. Also, every weak-solid edge is a weak-dotted edge.*

Proof. We rule out all alternative cases one by one. In particular we prove:

(i) *A strong-solid edge cannot be weak-dotted.* If an edge (i,j) is strong-solid then it cannot be adjacent to another solid edge (weak or strong). Therefore, using Lemma D.8 none of adjacent edges to (i,j) are dotted. However, if (i,j) is weak-dotted by Lemma D.3 it is adjacent to at least one other weak-dotted edge (since at least one of γ_i and γ_j is positive) which is a contradiction. Thus (i,j) cannot be weak-dotted.

(ii) *A strong-solid edge cannot be non-dotted.* Similar to (i), if an edge (i,j) is strong-solid it cannot be adjacent to dotted edges. Now, if (i,j) is non-dotted then $\gamma_i = \gamma_j = 0$ using Lemma D.4. Hence $w_{ij} < \gamma_i + \gamma_j = 0$ which is contradiction since we assumed all weights are positive.

(iii) *A weak-solid edge cannot be strong-dotted.* Assume, (i_1,i_2) is weak-solid and strong-dotted. Then we can show an optimum to LP (1.2) can be improved which is a contradiction. The proof is very similar to proof of Lemma D.8. Since (i_1,i_2) is weak-solid, there is a half-integral matching $\underline{x}^*$ that is optimum to LP and puts a mass 1/2 or 0 on (i_1,i_2). Then either there is an adjacent $\underline{x}^*$-solid edge (i_2,i_3) or an adjacent $\underline{x}^*$-solid edge (i_0,i_1) with mass at least 1/2 or we stop. In the latter case, increasing the value of $x^*_{i_1 i_2}$ increases $\sum_{e\in E} w_e x_e^*$ while keeping it LP feasible which is a contradiction. Otherwise, by strong-dotted assumption on (i_1,i_2) ((i_0,i_1)), the new edge (i_2,i_3) is not dotted. Now we select a dotted edge (i_3,i_4) if it exists (otherwise we stop and in that case $\gamma_{i_3} = 0$). This process is repeated as in proof of Lemma D.8 in both directions to obtain an alternating path $P = (i_{-k}, \ldots, i_{-1}, i_0, i_1, i_2, \ldots, i_\ell)$ with all odd edges being dotted with $\underline{x}^*$ value at most 1/2 and all even edges being $\underline{x}^*$-solid with mass at least 1/2. We discuss the case of P being a simple path (not intersecting itself) here, and other cases: cycle, bicycle and blossom can be treated similar to path as in proof of Lemma D.8.

Construct LP solution $\underline{x}'$ that is equal to $\underline{x}^*$ outside of P and inside it satisfies $x'_e = x_e^* + 1/2$ if e is an odd edge that is $e = (i_{2k-1,i_{2k}})$, and $x'_e = x_e^* - 1/2$ when e is an even edge that is $e = (i_{2k,i_{2k+1}})$. It is easy to see that $\underline{x}'$ is a feasible LP solution. And since for all edges (i_j, i_{j+1}) we have $\gamma_{i_j} + \gamma_{i_{j+1}} \ge w_{i_j i_{j+1}}$ and on dotted edges we have equality $\gamma_{i_j} + \gamma_{i_{j+1}} = w_{i_j i_{j+1}}$ then $\sum_{e\in E} w_e x_e^* - \sum_{e\in E} w_e x'_e = \frac{w(P_{\text{dotted}}) - w(P_{\text{solid}})}{2} \ge \frac{\gamma_{i_2} + \gamma_{i_3} - w_{i_2 i_3}}{2} > 0$ where the last inequality follows from the fact that (i_2,i_3) is not-dotted. Hence we reach a contradiction.

(iv) *A weak-solid edge cannot be non-dotted.* Assume, (i_1,i_2) is weak-solid and non-dotted. Similar to (iii) we can show the best solution to LP (1.2) can be improved which is a contradiction. Since (i_1,i_2) is weak-solid we can choose a half-integral $\underline{x}^*$ that puts a mass at least 1/2 on (i_1,i_2). Also, this time the alternation in P is the opposite of (iii). That is we choose (i_2,i_3) to be dotted (if it does not exist $\gamma_{i_2} = 0$ and we stop.) The solution $\underline{x}'$ is constructed as before: equal to $\underline{x}^*$ outside of P, $x'_e = x_e^* + 1/2$ if e is odd and $x'_e = x_e^* - 1/2$ if it is even. Hence, $\sum_{e\in E} w_e x_e^* - \sum_{e\in E} w_e x'_e \ge \frac{\gamma_{i_1} + \gamma_{i_2} - w_{i_1 i_2}}{2} > 0$, using the non-dotted assumption on (i_1,i_2). Hence, we obtain another contradiction.

Lemma D.10. *$\underline{\gamma}$ is an optimum for the dual problem (1.3)*

Proof. Lemma D.1 guarantees feasibility. Optimality follows from lemmas D.4, D.8 and D.9 as follows. Take any optimum half integral matching $\underline{x}^*$ to LP. Now using Lemma D.9: $\sum_v \gamma_v = \sum_{e\in E} w_e x_e^*$ which finishes the proof.

THEOREM D.1. *Let $\mathcal{BALOPT}$ be the set of optima of the dual problem (1.3) satisfying the unmatched balance property, Eq. (1.1), at every edge. If $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ is a fixed point of the natural dynamics then $\underline{\gamma} \in \mathcal{BALOPT}$. Conversely, for every $\underline{\gamma}_{\rm BO} \in \mathcal{BALOPT}$, there is a unique fixed point of the natural dynamics with $\underline{\gamma} = \underline{\gamma}_{\rm BO}$.*

Proof. The direct implication is immediate from Lemmas D.7 and D.10. The converse proof here follows the same steps as for Theorem 1, proved in Section 3. Instead of separately analyzing the cases $(ij) \in M$ and $(ij) \notin M$, we study the cases $\gamma_i + \gamma_j = w_{ij}$ and $\gamma_i + \gamma_j > w_{ij}$.

E ϵ-fixed point properties: Proof of Theorem 1.4

In this section we prove Theorem 1.4, stated in Section 1.3. In this section we assume that $\underline{\alpha}$ is an ϵ-fixed point with corresponding offers $\underline{m}$ and earnings $\underline{\gamma}$. That is, for all i, j

$$\epsilon \geq \left|\alpha_{i\backslash j} - \max_{k\in\partial i\backslash j} m_{k\to i}\right|,$$

$$m_{i\to j} = (w_{ij} - \alpha_{i\backslash j})_+ - \frac{(w_{ij} - \alpha_{i\backslash j} - \alpha_{j\backslash i})_+}{2},$$

$$\gamma_i = \max_{k\in\partial i} m_{k\to i}.$$

DEFINITION E.1. *An edge (ij) is called δ-dotted ($\delta \geq 0$) if $\gamma_i + \gamma_j \leq w_{ij} + \delta$.*

LEMMA E.1. *For all edge $(ij) \in E$ and all $\delta, \delta_1, \delta_2 \in \mathbb{R}$ the following hold:*

(a) If (ij) is δ-dotted then $\mathcal{S}urp_{ij} \geq -(2\epsilon + \delta)$.

(b) If $\mathcal{S}urp_{ij} \geq -\delta$ then $m_{i\to j} \geq \gamma_j - (\epsilon + \delta)$ and $m_{j\to i} \geq \gamma_i - (\epsilon + \delta)$.

(c) If $m_{i\to j} \geq \gamma_j - \delta_1$ and $m_{j\to i} \geq \gamma_i - \delta_2$ then (ij) is $(\delta_1 + \delta_2)$-dotted.

(d) If $\gamma_i - \delta \leq m_{j\to i}$ and $\gamma_j > m_{i\to j} + 2\epsilon + \delta$ then $\gamma_i = 0$.

(e) If $\gamma_i > 0$ and $m_{j\to i} \geq \gamma_i - \delta$ then (ij) is $(2\delta + 2\epsilon)$-dotted.

(f) For $\gamma_i, \gamma_j > 0$, $m_{j\to i} \leq \alpha_{i\backslash j} + \delta$ if and only if $m_{i\to j} \leq \alpha_{j\backslash i} + \delta$.

(h) For all (ij), $|m_{i\to j} - (w_{ij} - \gamma_i)_+| \leq \epsilon$.

(i) For all (ij), $\gamma_i - (w_{ij} - \gamma_j)_+ \geq -\epsilon$ and $\gamma_i + \gamma_j \geq w_{ij} - \epsilon$.

(j) For all i, if $\gamma_i > 0$ then there is at least a 2ϵ-dotted edge attached to i.

Proof. (a) Since $\underline{\alpha}$ is ϵ-fixed point, $\gamma_i \geq m_{i\to j} - \epsilon$ and $\gamma_j \geq m_{j\to i} - \epsilon$. Therefore, $\mathcal{S}\text{urp}_{ij} = w_{ij} - m_{i\to j} - m_{j\to i} \geq w_{ij} - \gamma_i - \gamma_j - (2\epsilon) \geq -(2\epsilon + \delta)$.

(b) First consider the case $\mathcal{S}\text{urp}_{ij} \leq 0$. Then, $m_{i\to j} = (w_{ij} - \alpha_{i\backslash j})_+ \geq w_{ij} - \alpha_{i\backslash j} \geq \alpha_{j\backslash i} - \delta \geq \max_{\ell\in\partial j\backslash i}(m_{\ell\to j}) - \delta - \epsilon$, which yields $m_{i\to j} \geq \gamma_j - (\epsilon + \delta)$. The proof of $m_{j\to i} \geq \gamma_i - (\epsilon + \delta)$ is similar.

For the case $\mathcal{S}\text{urp}_{ij} > 0$, $m_{i\to j} = \frac{w_{ij} - \alpha_{i\backslash j} + \alpha_{j\backslash i}}{2} = \frac{\mathcal{S}\text{urp}_{ij}}{2} + \alpha_{i\backslash j} \geq \max(\frac{-\delta}{2}, 0) + \max_{\ell\in\partial j\backslash i}(m_{\ell\to j}) - \epsilon$, and the rest follows as above.

(c) Note that $\gamma_i + \gamma_j \leq m_{i\to j} + m_{j\to i} + \delta_1 + \delta_2$. If $\mathcal{S}\text{urp}_{ij} \geq 0$ then the result follows from $m_{i\to j} + m_{j\to i} = w_{ij}$. For $\mathcal{S}\text{urp}_{ij} < 0$ the result follows from $m_{i\to j} + m_{j\to i} \leq \max[(w_{ij} - \alpha_{i\backslash j})_+, (w_{ij} - \alpha_{j\backslash i})_+, 2w_{ij} - \alpha_{i\backslash j} - \alpha_{j\backslash i}] \leq w_{ij}$.

(d) We need to show that when $\gamma_i \leq m_{j\to i} + \delta$ and $\gamma_j > m_{i\to j} + 2\epsilon + \delta$ then $\gamma_i = 0$. From part (b) that was just shown, the surplus should satisfy $\mathcal{S}\text{urp}_{ij} < -(\epsilon + \delta)$. On the other hand $\alpha_{i\backslash j} - \epsilon \leq \max_{k\in\partial i\backslash j}(m_{k\to i}) \leq \gamma_i \leq m_{j\to i} + \delta \leq (w_{ij} - \alpha_{i\backslash j})_+ + \delta$. Now, if $\gamma_i > 0$ then $w_{ij} - \alpha_{i\backslash j} > 0$ which gives $\alpha_{i\backslash j} - \epsilon \leq w_{ij} - \alpha_{i\backslash j} + \delta$. This is equivalent to $\mathcal{S}\text{urp}_{ij} \geq -(\epsilon + \delta)$ which is a contradiction. Hence $\gamma_i = 0$.

(e) Using part (d) we should have $m_{i\to j} \geq \gamma_j - (2\epsilon + \delta)$. Now applying part ($c$) the result follows.

(f) If $\mathcal{S}\text{urp}_{ij} \geq 0$ then $\frac{w_{ij} - \alpha_{j\backslash i} + \alpha_{i\backslash j}}{2} = m_{j\to i} \leq \alpha_{i\backslash j} + \delta$. This inequality is equivalent to $m_{i\to j} = \frac{w_{ij} - \alpha_{i\backslash j} + \alpha_{j\backslash i}}{2} \leq \alpha_{j\backslash i} + \delta$, which proves the result. If $\mathcal{S}\text{urp}_{ij} < 0$ then $w_{ij} - \alpha_{j\backslash i} \leq (w_{ij} - \alpha_{j\backslash i})_+ \leq \alpha_{i\backslash j} + \delta$. This is equivalent to $w_{ij} - \alpha_{i\backslash j} \leq \alpha_{j\backslash i} + \delta$ which yields the result.

(h) If $\mathcal{S}\text{urp}_{ij} \geq 0$ then by part (b), $m_{i\to j} + \epsilon \geq \gamma_j$ and $m_{j\to i} + \epsilon \geq \gamma_i$. Therefore, using $\gamma_j \geq m_{i\to j}$, $\gamma_i \geq m_{j\to i}$ and $m_{j\to i} + m_{i\to j} = w_{ij}$ we have, $m_{i\to j} \geq w_{ij} - m_{j\to i} \geq w_{ij} - \gamma_i \geq w_{ij} - m_{j\to i} - \epsilon \geq m_{i\to j} - \epsilon$, which gives the result.

If $\mathcal{S}\text{urp}_{ij} < 0$ then $m_{i\to j} = (w_{ij} - \alpha_{i\backslash j})_+ < \alpha_{j\backslash i}$ this gives $\gamma_j - \epsilon < \alpha_{j\backslash i}$. On the other hand $\alpha_{j\backslash i} \leq \gamma_j + \epsilon$ holds. Similarly, $\gamma_i + \epsilon \geq \alpha_{i\backslash j} \geq \gamma_i - \epsilon$ that leads to $|(w_{ij} - \alpha_{i\backslash j})_+ - (w_{ij} - \gamma_i)_+| \leq \epsilon$. Hence, the result follows from $m_{i\to j} = (w_{ij} - \alpha_{i\backslash j})_+$.

(i) Using part (h), $m_{j\to i} + \epsilon \geq (w_{ij} - \gamma_j)_+$. Now result follows using $\gamma_i \geq m_{j\to i}$.

(j) There is at least one neighbor $j \in \partial i$ that sends the maximum offer $m_{j\to i} = \gamma_i$. Using part (d) we should have $m_{i\to j} \geq \gamma_j - 2\epsilon$ and now the result follows from part (c).

LEMMA E.2. *For any edge $(ij) \in E$ the earnings estimate $\underline{\gamma}$ satisfies 6ϵ-balanced property (i.e., Eq. (1.9) holds for 6ϵ instead of ϵ).*

Proof. Using Lemma E.1(h), $\alpha_{i\backslash j} - 2\epsilon \leq$

$\max_{k\in\partial i\setminus j}(m_{k\to i}) - \epsilon \le \max_{k\in\partial i\setminus j}[(w_{ik}-\gamma_k)_+] \le \max_{k\in\partial i\setminus j}(m_{k\to i}) + \epsilon \le \alpha_{i\setminus j} + 2\epsilon$, or

$$\left|\max_{k\in\partial i\setminus j}[(w_{ik}-\gamma_k)_+] - \alpha_{i\setminus j}\right| \le 2\epsilon \tag{E.11}$$

Now, if $\mathcal{S}\mathrm{urp}_{ij} \le 0$ then $m_{j\to i} = (w_{ij} - \alpha_{j\setminus i})_+ \le \alpha_{i\setminus j}$ which gives $|\gamma_i - \alpha_{i\setminus j}| \le \epsilon$ or, $\left|\gamma_i - \max_{k\in\partial i\setminus j}[(w_{ik}-\gamma_k)_+]\right| \le 3\epsilon$. Therefore, 6ϵ-balance property holds.

And if $\mathcal{S}\mathrm{urp}_{ij} > 0$, by Lemma E.1(b) we have $m_{j\to i} + \epsilon \ge \gamma_i$. Hence, $\frac{\mathcal{S}\mathrm{urp}_{ij}}{2} + \epsilon = m_{j\to i} - \alpha_{i\setminus j} + \epsilon \ge \gamma_i - \alpha_{i\setminus j} \ge m_{j\to i} - \alpha_{i\setminus j} = \frac{\mathcal{S}\mathrm{urp}_{ij}}{2}$. Same bound holds for $\gamma_j - \alpha_{j\setminus i}$ by symmetry. Therefore, using Eq. (E.11), $|\gamma_i - \max_{k\in\partial i\setminus j}[(w_{ik}-\gamma_k)_+]|$ and $|\gamma_j - \max_{\ell\in\partial j\setminus i}[(w_{j\ell}-\gamma_\ell)_+]|$ are within $3\epsilon \le 6\epsilon$ of each other.

LEMMA E.3. *If (ij) is δ-dotted for $k \in \partial i\setminus j$ and if $\gamma_k > \max(\delta,\epsilon) + 6\epsilon$, then there exists $r \in \partial k\setminus i$ such that (rk) is $(\max(\delta,\epsilon)+6\epsilon)$-dotted.*

Proof. Using, $\gamma_i + \gamma_j \le w_{ij} + \delta$ and Lemma E.1(i),

$$-\epsilon \le \gamma_i - \max_{s\in\partial i\setminus k}[(w_{is}-\gamma_s)_+] \le \gamma_i - (w_{ij}-\gamma_j)_+ \le \delta.$$

Therefore, $|\gamma_i - \max_{s\in\partial i\setminus k}[(w_{is}-\gamma_s)_+]| \le \max(\delta,\epsilon)$ which combined with Lemma E.2 gives

$$|\gamma_k - \max_{r\in\partial k\setminus i}[(w_{rk}-\gamma_r)_+]| \le \max(\delta,\epsilon) + 6\epsilon.$$

This fact and $\gamma_k > \max(\delta,\epsilon)+6\epsilon$, show that there exists an edge $r \in \partial k\setminus i$ with $|\gamma_k - (w_{rk}-\gamma_r)_+| \le \max(\delta,\epsilon)+6\epsilon$ and the result follows.

LEMMA E.4. *A non-solid edge cannot be a δ-dotted edge for $\delta \le 4\epsilon$.*

Note that this Lemma holds even for the more general case of M^* being non-integral.

The proof is a more complex version of proof of Lemma D.8. Recall the notion of alternating path from that proof.

Also, consider $\underline{x}^*$ and $\underline{y}^*$ that are optimum solutions for the LP and its dual, (1.2) and (1.3). Also recall that by complementary slackness conditions , for all solid edges the equality $y_i^* + y_j^* = w_{ij}$ holds. Moreover, any node $v \in V$ is adjacent to a solid edge iff $y_v^* > 0$.

Proof. [Proof of Lemma E.4] We need to consider two cases:

Case (I). Assume that the optimum LP solution $\underline{x}^*$ is integral (having a tight LP). Now assume the contrary: take (i_1,i_2) that is a non-solid edge but it is δ-dotted. Consider an endpoint of (i_1,i_2). For example take i_2. Either there is a solid edge attached to i_2 or not. If there is not, we stop. Otherwise, assume (i_2,i_3) is a solid edge. Using Lemma E.3, either $\gamma_{i_3} > 10\epsilon$ or there is a 10ϵ-dotted edge (i_3,i_4) connected to i_3. Now, depending on whether i_4 has (has not) an adjacent solid edge we continue (stop) the construction. Similar procedure could be done by starting at i_1 instead of i_2. Therefore, we obtain an alternating path $P = (i_{-k},\dots,i_{-1},i_0,i_1,i_2,\dots,i_\ell)$ with each (i_{2k},i_{2k+1}) being $(6k+4)\epsilon$-dotted and all $(i_{2k-1},i_{2k}))$ being solid. Using the same argument as in [6] one can show that one of the following four scenarios occur.

Path: Before P intersects itself, both end-points of the path stop. At each end of the path, either the last edge is solid (then $\gamma_v < (3n+4)\epsilon$ for the last node v) or the last edge is a $(3n+4)$-dotted edge with no solid edge attached to v. Now consider a new solution $\underline{x}'$ to LP (1.2) by $x'_e = x^*_e$ if $e \notin P$ and $x'_e = 1 - x^*_e$ if $e \in P$. It is easy to see that $\underline{x}'$ is a feasible LP solution at all points $v \notin P$ and also for internal vertices of P. The only nontrivial case is when $v = i_{-k}$ (or $v = i_\ell$) and the edge (i_{-k}, i_{-k+1}) (or $(i_{\ell-1}, i_\ell)$) is $(3n+4)\epsilon$-dotted. In both of these cases, by construction no solid edge is attached to v outside of P so making any change inside of P is safe. Now denote the weight of all solid (remaining) edges of P by $w(P_{\mathrm{solid}})$ $(w(P_{\mathrm{dotted}}))$. Hence, $\sum_{e\in E} w_e x^*_e - \sum_{e\in E} w_e x'_e = w(P_{\mathrm{solid}}) - w(P_{\mathrm{dotted}})$.

But $w(P_{\mathrm{dotted}}) + (3n^2+16n)\epsilon/4 \ge \sum_{v\in P}\gamma_v$. Moreover, from Lemma E.1(i), $\gamma_i + \gamma_j \ge w_{ij} - \epsilon$ for all $(ij) \in P$ which gives $w(P_{\mathrm{solid}}) \le \sum_{v\in P}\gamma_v + n\epsilon/2$. Now $\sum_{e\in E} w_e x^*_e - \sum_{e\in E} w_e x'_e = w(P_{\mathrm{solid}}) - w(P_{\mathrm{dotted}})$ yields $w_e x^*_e - \sum_{e\in E} w_e x'_e \le (3n^2+18n)\epsilon/4 \le n(n+5)\epsilon$. For $\epsilon < g/(6n^2)$ This contradicts the tightness of LP relaxation (1.2) since $x'_e \ne x^*_e$ holds at least for $e = (i_1,i_2)$.

Cycle: P intersects itself and will contain an even cycle C_{2s}. This case can be handled very similar to the path by defining $x'_e = x^*_e$ if $e \notin C_{2s}$ and $x'_e = 1 - x^*_e$ if $e \in C_{2s}$. The proof is even simpler since the extra check for the boundary condition is not necessary.

Blossom: P intersects itself and will contain an odd cycle C_{2s+1} with a path (stem) P' attached to the cycle at point u. In this case let $x'_e = x^*_e$ if $e \notin P' \cup C_{2s+1}$, and $x'_e = 1 - x^*_e$ if $e \in P'$, and $x'_e = \frac{1}{2}$ if $e \in C_{2s+1}$. From here, we drop the subindex $2s+1$ to simplify the notation. Since the cycle has odd length, both neighbors of u in C have to be dotted. Therefore,

$$\sum_{e\in E} w_e x^*_e - \sum_{e\in E} w_e x'_e = w(P'_{\mathrm{solid}}) + w(C_{\mathrm{solid}}) - w(P'_{\mathrm{dotted}}) - \frac{w(C_{\mathrm{dotted}}) + w(C_{\mathrm{solid}})}{2},$$

which is equal to $w(P'_{\text{solid}}) + \frac{w(C_{\text{solid}})}{2} - w(P'_{\text{dotted}}) - \frac{w(C_{\text{dotted}})}{2}$ and is less than

$$\sum_{v\in P'} \gamma_v + \lceil \frac{|P|}{2} \rceil \epsilon + \frac{\sum_{v\in C}\gamma_v - \gamma_u}{2} + s\epsilon - \sum_{v\in P'} \gamma_v$$
$$+\gamma_u + (\frac{3|P|^2+16|P|}{4})\epsilon - \frac{\sum_{v\in C}\gamma_v + \gamma_u}{2} + (\frac{3s^2+16s}{4})\epsilon\,.$$

But the last term is at most $n(n+5)\epsilon$ which is again a contradiction.

Bicycle: P intersects itself at least twice and will contain two odd cycles C_{2s+1} and $C'_{2s'+1}$ with a path (stem) P' that is connecting them. Very similar to Blossom, let $x'_e = x^*_e$ if $e \notin P' \cup C \cup C'$, $x'_e = 1 - x^*_e$ if $e \in P'$, and $x'_e = \frac{1}{2}$ if $e \in C \cup C'$. The proof follows similar to the case of blossom.

Case (II). Assume that the optimum LP solution $\underline{x}^*$ is not necessarily integral.

Everything is similar to Case (I) but the algebraic treatments are slightly different. Some edges e in P can be $\frac{1}{2}$-solid ($x^*_e = \frac{1}{2}$). In particular some of the odd edges (dotted edges) of P can now be $\frac{1}{2}$-solid. But the subset of $\frac{1}{2}$-solid edges of P can be only sub-paths of odd length in P. On each such sub-path defining $\underline{x}' = 1 - \underline{x}^*$ means we are not affecting $\underline{x}^*$. Therefore, all of the algebraic calculations should be considered on those sub-paths of P that have no $\frac{1}{2}$-solid edge which means both of their boundary edges are dotted.

Path: Define $\underline{x}'$ as in Case (I). Using the discussion above, let $P^{(1)}, \ldots, P_{(r)}$ be disjoint sub-paths of P that have no $\frac{1}{2}$-solid edge. Thus, $\sum_{e\in E} w_e x^*_e - \sum_{e\in E} w_e x'_e = \sum_{i=1}^{r} \big[w(P^{(i)}_{\text{solid}}) - w(P^{(i)}_{\text{dotted}})\big]$. Since in each $P^{(i)}$ the two boundary edges are dotted, $w(P^{(i)}_{\text{solid}}) \le \sum_{v\in P^{(i)}} \gamma_v + |P^{(i)}|\epsilon/2$ and $\sum_{v\in P^{(i)}} \gamma_v \le w(P^{(i)}_{\text{dotted}}) + (3|P^{(i)}|^2 + 16|P^{(i)}|)\epsilon/4$. The rest can be done as in Case (I).

Cycle, Blossom, Bicycle: These cases can be done using the same method of breaking the path and cycles into sub-paths $P^{(i)}$ and following the case of path.

The direct part of Theorem 1.4 follows from the next lemma.

LEMMA E.5. *$\underline{\alpha}$ induces the matching M^*.*

Proof. From Lemma E.4 it follows that the set of 2ϵ-dotted edges is a subset of the solid edges. In particular, when the optimum matching M^* is integral, no node can be adjacent to more than one 2ϵ-dotted edges. If we define a $\underline{x}'$ to be zero on all edges and $x'_e = 1$ for all 2ϵ-dotted edges (ij) with $\gamma_i + \gamma_j > 0$. Then clearly $\underline{x}'$ is feasible to (1.2). On the other hand, using the definition of 2ϵ-dotted for all e' with $x_{e'} = 1$, and Lemma E.1(j) that each node with $\gamma_i > 0$ is adjacent to at least one 2ϵ-dotted edge we can write $\sum_{e\in E} w_e x'_e \ge \sum_{v\in V} \gamma_v - n\epsilon$. Separately, from Lemma E.1(i) we have $\sum_{v\in V} \gamma_v \ge \sum_{e\in E} w_e x^*_e - \frac{n\epsilon}{2}$, which shows that $\underline{x}'$ is also an optimum solution to (1.2) (when $\epsilon < g/(6n^2)$). From the uniqueness assumption on $\underline{x}^*$ we obtain that M^* is equal to the set of all 2ϵ-dotted edges with at least one endpoint having a positive earning estimate. We would like to show that for any such edge (ij), both earning estimates γ_i and γ_j are positive.

Assume the contrary, i.e., without loss of generality $\gamma_i = 0$. Then, $\mathcal{S}\text{urp}_{ij} \le 0$ and $0 = m_{j\to i} = (w_{ij} - \alpha_{j\setminus i})_+$ that gives $\alpha_{j\setminus i} \ge w_{ij}$ or

$$m_{\ell\to j} \ge \alpha_{j\setminus i} - \epsilon \ge w_{ij} - \epsilon \ge (w_{ij} - \alpha_{i\setminus j})_+ - \epsilon = \gamma_j - \epsilon.$$

for some $\ell \in \partial j\setminus i$. Now using Lemma E.1(e) the edge $(j\ell)$ is 4ϵ-dotted which contradicts Lemma E.4.

Finally, the endpoints of the matched edges provide each other their unique best offers. This latter follows from the fact that each node with $\gamma_i > 0$ receives an offer equal to γ_i and the edge corresponding to that offer has to be 2ϵ-dotted using Lemma E.1(d). The nodes with no positive offer $\gamma_i = 0$ are unmatched in M^* as well.

Proof of Theorem 1.4.

Proof. For any $\epsilon < g/(6n^2)$, an ϵ-fixed point induces the matching M^* using Lemma E.5. Additionally, the earning vector $\underline{\gamma}$ is (6ϵ)-balanced using Lemma E.2. Next we show that $(\underline{\gamma}, M^*)$ is a stable trade outcome.

LEMMA E.6. *The earnings estimates $\underline{\gamma}$ is an optimum solution to the dual* (1.3). *In particular the pair $(\underline{\gamma}, M^*)$ is a stable trade outcome.*

Proof. Using Lemma E.4, we can show that for any non-solid edge (ij), stability holds, i.e. $\gamma_i + \gamma_j \ge w_{ij}$.

Now let (i,j) be a solid edge. Then i and j are sending each other their best offers. If $\mathcal{S}\text{urp}_{ij} \ge 0$ we are done using $\gamma_i + \gamma_j = m_{j\to i} + m_{i\to j} = \frac{w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i}}{2} + \frac{w_{ij} - \alpha_{j\setminus i} + \alpha_{i\setminus j}}{2} = w_{ij}$. And if $\mathcal{S}\text{urp}_{ij} < 0$ then $\gamma_i = m_{j\to i} = (w_{ij} - \alpha_{j\setminus i})_+ \le \alpha_{i\setminus j}$. Similarly, $\gamma_j \le \alpha_{j\setminus i}$. This means there exist $k \in \partial i\setminus j$ with $m_{k\to i} \ge \alpha_{i\setminus j} - \epsilon \ge \gamma_i - \epsilon$. But, from Lemma E.1(e) the edge (ik) would become 4ϵ-dotted which is a contradiction.

The converse of Theorem 1.4 is trivial since any ϵ-NB solution $(M', \underline{\gamma}_{\text{NB}})$ is stable and produces a trade outcome by definition, hence it is a dual optimal solution which means $M' = M^*$.

F Relationship to Belief Propagation

F.1 Max product BP It is known that max-product belief propagation (BP), for maximum weight matching correctly finds the MWM iff the LP relaxation (1.2) admits a unique integral optimum (see, e.g. [5], [6], [26]).[1] The analysis leading to this result involves the 'computation tree' of the BP updates, and shows that convergence to the correct MWM occurs in $O(n/g)$ iterations.

It turns out that the natural dynamics proposed in this work is closely related to BP. Consider (1.4). If we drop the second 'surplus division' term we obtain

$$m^t_{i\to j} \stackrel{\text{BP}}{=} (w_{ij} - \alpha^t_{i\backslash j})_+ \,. \tag{F.12}$$

If we use this new definition in the update rule (1.6), we obtain damped belief propagation updates for maximum weight matching.[2] Thus, the natural dynamics only differs the standard BP in that it includes an extra term that reduces 'offers' by half the zero-thresholded edge surplus! Moreover, note that T defined by (F.12) and (2.12) is non-expansive. It is worth noting that the BP updates also have a bargaining interpretation [2].

Now consider the following generalization of (1.4), parameterized by β.

$$m^t_{i\to j} \stackrel{\text{BP-}\beta}{=\!=} (w_{ij} - \alpha^t_{i\backslash j})_+ - \beta(w_{ij} - \alpha^t_{i\backslash j} - \alpha^t_{j\backslash i})_+ \,. \tag{F.13}$$

We retain the update rule (1.6). With $\beta = 0$ we obtain BP, and with $\beta = 1/2$ we obtain the natural dynamics. Again, T defined by (F.13) and (2.12) is non-expansive, so we are guaranteed fast convergence. We show, in fact, that for $\beta \in [0, 1)$, BP-β *successfully finds the maximum weight matching in* $O(n^4/\left((1-\beta)g\right)^2)$ *iterations* if the LP (1.2) has a unique optimum that is integral.

Thus, we unify our understanding of why both BP and natural dynamics find the maximum weight matching.

THEOREM F.1. *Suppose the LP (1.2) has a unique optimum and this is integral, corresponding to matching M^*. Let $\underline{\alpha}$ be a fixed point of BP-β, for $\beta \in [0, 1)$. Then $\underline{\alpha}$ induces matching M^*.*

Here the meaning of 'induces a matching' is as in Definition 1.1.

Proof. [Sketch of proof] The key step is appropriately defining $\underline{\gamma}$. Sort the offers received by node i in descending order. Denote by $\mu_i(\ell)$ the ℓ-th offer in the list ($\mu^t_i(\ell) = 0$ if $\ell > |\partial i|$). Thus, $\mu_i(1)$ is the best offer received and $\mu_i(2)$ is the next best offer. Then we define

$$\gamma_i \stackrel{\text{BP-}\beta}{=\!=} \frac{1}{2(1-\beta)}\mu_i(1) + \left(1 - \frac{1}{2(1-\beta)}\right)\mu_i(2) \tag{F.14}$$

[1] This algorithm forms the first phase of the local algorithm for finding balanced outcomes proposed in [2].

[2] Typically, the version of BP studied is the one without damping.

Now, the proof mirrors the proof of the direct part of Theorem 1.1. Define solid and dotted (weak and strong) edges as before. Check that at a fixed point for $(ij) \in E$, $\gamma_i + \gamma_j = w_{ij}$ if (ij) is dotted and that $\gamma_i + \gamma_j \ge w_{ij}$ otherwise. This enables us to show that Property 6 holds: An edge is 1-solid(1/2-solid) iff it is strongly(weakly) dotted (Proofs of Lemmas D.4, D.8 and D.9 go through verbatim). The theorem follows.

Further, we can extend our characterization to approximate fixed points of BP-(β) for $\beta \in [0, 1)$, mirroring the steps followed for the natural dynamics (see Appendix E). We can show that for $\epsilon \le \epsilon^* = O\left((1-\beta)g/n^2\right)$, if $\underline{\alpha}$ is an ϵ-FP under BP-β, then it induces the matching M. We already know that Theorem 1.3 holds for BP-β. As such, BP-β finds M in $O\left(n^4/\left((1-\beta)g\right)^2\right)$ iterations.

F.2 Tree reweighted message passing We note that Eq. (1.4) differs from standard belief propagation in a key way: $m^t_{i\to j}$ depends on the message $\alpha^t_{j\backslash i}$ coming in the opposite direction, as well as on $\alpha^t_{i\backslash j}$. In standard belief propagation, $m^t_{i\to j}$ only depends on $\alpha^t_{i\backslash j}$ and not on $\alpha^t_{j\backslash i}$.

Looking more closely, we find that updates based on Eq. (1.4) bear a strong resemblance to the tree reweighted message passing updates constructed by Wainwright et al [31] to solve the problem of finding exact MAP estimates on loopy graphs. More precisely, we can show that the natural dynamics corresponds to the tree reweighted message passing updates ([31, Algorithm 1]) for maximum weight matching where the 'edge appearance probability' ρ_{ij} is replaced by the message dependent function

$$\rho^t_{ij} = \begin{cases} 1 & \text{if } \alpha^t_{i\backslash j} + \alpha^t_{j\backslash i} \ge w_{ij}\,, \\ \frac{1}{2} & \text{otherwise}\,. \end{cases} \tag{F.15}$$

G The Kleinberg-Tardos construction and the KT gap

Let G be an instance which admits at least one stable outcome, M^* be the corresponding matching (recall that this is a maximum weight matching), and consider the Kleinberg-Tardos (KT) procedure for finding a NB solution [19]. Any NB solution $\underline{\gamma}^*$ can be constructed by this procedure with appropriate choices at successive stages. At each stage, a linear program is solved with variables γ_i attached to node i. The linear program maximizes the minimum 'slack' of all unmatched edges and nodes, whose values have not yet been set (the slack of edge $(i, j) \notin M$ is $\gamma_i + \gamma_j - w_{ij}$).

At the first stage, the set of nodes that remain unmatched (i.e. are not part of M^*) is found, if such

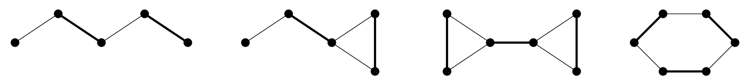

Figure 1: Examples of basic structures: path, blossom, bicycle, and cycle (matched edges in bold).

nodes exist. Call the set of unmatched nodes $\mathcal{C}_0$.

After this, at successive stages of the KT procedure, a sequence of structures $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ characterizing the LP optimum are found. We call this the *KT sequence*. Each such structure is a pair $\mathcal{C}_q = (V(\mathcal{C}_q), E(\mathcal{C}_q))$ with $V(\mathcal{C}_q) \subseteq V$, $E(\mathcal{C}_q) \subseteq E$. According to [19] $\mathcal{C}_q$ belongs to one of four topologies: alternating path, blossom, bicycle, alternating cycle (Figure 1). The q-th linear program determines the value of γ_i^* for $i \in V(\mathcal{C}_q)$. Further, one has the partition $E(\mathcal{C}_q) = E_1(\mathcal{C}_q) \cup E_2(\mathcal{C}_q)$ with $E_1(\mathcal{C}_q)$ consisting of all matching edges along which nodes in $V(\mathcal{C}_q)$ trade, and $E_2(\mathcal{C}_q)$ consists of edges (i,j) such that some $i \in V(\mathcal{C}_q)$ receives its second-best, positive offer from j.

The γ values for nodes on the limiting structure are uniquely determined if the structure is an alternating path, blossom or bicycle[3]. In case of an alternating cycle there is one degree of freedom – setting a value γ_i^* for one node $i \in \mathcal{C}_q$ fully determines the values at the other nodes.

We emphasize that, within the present definition, $\mathcal{C}_q$ is not necessarily a subgraph of G, in that it might contain an edge (i,j) but not both its endpoints. On the other hand, $V(\mathcal{C}_q)$ is always subset of the endpoints of $E(\mathcal{C}_q)$. We denote by $V_{\rm ext}(\mathcal{C}_q) \supseteq V(\mathcal{C}_q)$ the set of nodes formed by all the endpoints of edges in $E(\mathcal{C}_q)$.

[3] In [19] it is claimed that the γ values 'may not be fully determined' also in the case of bicycles. However it is not hard to prove that γ values are, in fact, uniquely determined in bicycles.

For all nodes $i \in V(\mathcal{C}_q)$ the second best offer is equal to $\gamma_i^* - \sigma_q$, where σ_q is the slack of the q-th structure. Therefore

$$\gamma_i^* + \gamma_j^* - w_{ij} = \begin{cases} 0 & \text{if } (i,j) \in E_1(\mathcal{C}_q), \\ \sigma_q & \text{if } (i,j) \in E_2(\mathcal{C}_q). \end{cases}$$

The slacks form an increasing sequence ($\sigma_1 \le \sigma_2 \le \ldots \le \sigma_k$).

Definition G.1. *We say that a unique Nash bargaining solution $\underline{\alpha}^*$ has a KT gap σ if*

$$\sigma \le \min\left\{\sigma_1;\, \sigma_2 - \sigma_1;\, \ldots;\, \sigma_k - \sigma_{k-1}\right\},$$

and if for each edge (i,j) such that $i, j \in V_{\rm ext}(\mathcal{C}_q)$ and $(i,j) \notin E(\mathcal{C}_q)$,

$$\gamma_i^* + \gamma_j^* - w_{ij} \ge \sigma_q + \sigma\,.$$

It is possible to prove that the positive gap condition is generic in the following sense. The set of all instances such that the NB solution is unique can be regarded as a subset $\mathsf{G} \subseteq [0, W]^{|E|}$ (W being the maximum edge weight). It turns out that G has dimension $|E|$ (i.e. the class of instances having unique NB solution is large) and that the subset of instances with gap $\sigma > 0$ is both *open and dense in* G.