THE ANALYTIC CAUCHY PROBLEM WITH SINGULAR DATA

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PREFACE

This work is one of the many which has been inspired by Leray's monumental Problème de Cauchy series of articles. In it, I treat a classical problem using rather old-fashioned methods. Indeed, Cauchy himself would have found little that was alien to him in this dissertation. In the analytic case, there seems to be no substitute for the classical method of finding a solution: construct an infinite series which formally solves the problem, then prove that it converges. Unfortunately, this method yields dull proofs -- expecially since I have included all the gruesome details. The results themselves are fairly simple to state, and I hope of sufficient interest to justify this effort. I have tried to organize the material so as to ease the burden of those compelled to examine all the detail, while still allowing the less compulsive reader to skim through the proofs. I apologize here for the use of the pedantic "we" in the writing. Somehow, the nature of the material seemed to demand it.

I will take this opportunity to informally describe the process by which I arrived at these results. In writing formal mathematics, one is usually very careful to obliterate the path which actually led him to the results. The course of my own wanderings may be of interest, especially to those readers familiar with the results on which my work is based.

The extension of Hamada's result comes from the method of constructing a series solution given by (10-11). It gives the required solution as a sum of solutions of the type found by Hamada. The key to convergence was the -1 inside the factorial expression in (5.6) of Mizohata's Proposition 3. (This proposition is essentially quoted in the Appendix.) It meant that the terms in the sum should get smaller. After some unsuccessful tries at using this key (one of them involved trying to extend Mizohata's result to allow the factorialed expression to be negative, and led to the convenient definition of $k!$ for $k < 0$ given in Section 11), I arrived at the method used in the proof of Lemma 13.4. The rest was a simple mass of detail, culminating in Lemma 13.5.

The extension to finitely ramified multi-valued functions was something of an afterthought. While not terribly exciting, it was a fairly easy generalization.

Unfortunately, I received Wagschal's paper too late for it to influence this dissertation. He has simplified the proof of Hamada's result, as well as generalizing it. Hopefully, his method will do the same for my results.

Theorem II resulted from a fairly direct application of the methods of Garding, Kotake and Leray. Theorem III was a serendipitous result which fell out of the process.

Theorem IV had its origins in an oversight, when I mistakenly thought that Hamada's method gave a solution which converged along the length of a characteristic surface. It didn't, but an even simpler method worked in the two-dimensional case. The generalization of the result to higher dimensions currently defies conjecture. The thing that makes the statement of the result easy in the two-dimensional case is the fact that a characteristic surface is just the range of a characteristic path.

The introduction of compact subsets in the definition of the influence domain was made to patch a hole in my original "proof". I do not know if it is necessary.

I wish to take this opportunity to thank Professor Takeshi Kotake, who guided me in this work. I also want to thank Professor Richard Palais for his help in preparing this dissertation, as well as for his many other contributions to my education.

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INTRODUCTION

We will consider the Cauchy problem for an analytic linear partial differential operator. Leray [6] has conjectured that the singularities of the solution lie in the characteristic surfaces which either emanate from the singularities of the initial date, or are tangent to the initial surface. In other words, singularities of the solution can arise in the following two ways:

(i) If the initial data are singular, then the solution may have singularities lying in the characteristic surfaces emanating from the singularities of the initial data.

(ii) If the initial surface has characteristic points, then the solution may have singularities in the characteristic surfaces tangent to the initial surface at these points.

Leray considered case (ii), with analytic initial date. Garding, Kotake and Leray [4] simplified his methods, and showed that in this case the solution can be locally uniformized. Uniformizing the solution u means finding a mapping σ such that $u \circ \sigma$ is analytic. Moreover, they constructed the uniformizing mapping σ so that, with certain restrictions on the set of characteristic points, this showed that the solution u is an algebraic function, finitely ramified about the tangent characteristic surface.

Hamada [5] proved (i) in the neighborhood of a non-characteristic point -- under the assumptions that the set of singularities is a submanifold, and that the characteristic surfaces emanating from it have multiplicity one and are not tangent to one another. He showed that if the initial data have only polar singularities, then the solution also just has polar singularities.

We will extend Hamada's result to cover some cases when the characteristic surfaces which emanate from the set of singularities have constant multiplicities. More precisely, we assume that the principal part of the operator can be factored into the product of powers of operators. The characteristic surfaces of these operators must all be distinct, non-tangent to one another, and of multiplicity one. In this case, the solution may have essential singularities even though the initial date have only poles. We will also extend the results to include multi-valued singular initial data which are finitely ramified about a submanifold.

The method of proof is an extension of the one used by Hamada. It involves constructing a formal series solution consisting of sums of powers of characteristic functions, then proving that the series converges. The construction requires solving a sequence of Cauchy problems for first order partial differential operators, then proving convergence using bounds for the solutions given by results of Mizohata [8].

We will also consider cases (i) and (ii) together. That is, we suppose the initial data to be singular at a characteristic point. We then show that the solution u is "uniformized" by the mapping σ constructed in [4], so that $u \circ \sigma$ has singularities along surfaces emanating from the singularities of the initial data. Moreover, if the initial data has poles of sufficiently low order which lie in the set of characteristic points, then $u \circ \sigma$ is analytic. These results are proved by showing that $u \circ \sigma$ is the solution of a Cauchy problem with no characteristic points.

Finally, for the two-dimensional case we give a global result which is closely related to (i). Given a domain V in the initial plane, we construct a domain $\mathcal{J}(V)$ in the two-dimensional space such that the solution is analytic on $\mathcal{J}(V)$ whenever the initial data is analytic on V. We define $\mathcal{J}(V)$ so that it contains no point on any characteristic surface emanating from a point outside V. (Hence, singularities of the initial data cannot propagate along characteristic surfaces into $\mathcal{J}(V)$.) The proof involves constructing a formal series solution, then proving convergence using a simple extension of Mizohata's results.

Precise statements of our results, as well as definitions of the terms used above, are given in Chapter I. The proofs are contained in the subsequent chapters.

I. DEFINITIONS AND RESULTS

1. Preliminary Notations

We let $\mathbb{C}$ denote the field of complex numbers, and $\mathbb{R}$ the real number field. The vector space of n-tuples of complex numbers is denoted by $\mathbb{C}^n$, and similarly $\mathbb{R}^n$ denotes the space of n-tuples of real numbers, $n \geqq 1$. We will identify $\mathbb{C}^n$ with the subset $\{(0, y^1, \ldots, y^n) : y^i \in \mathbb{C}\}$ of $\mathbb{C}^{n+1}$. Similarly, $\mathbb{C}^{n-1}$ will be identified with the subset $\{(0, y^2, \ldots, y^n) : y^i \in \mathbb{C}\}$ of $\mathbb{C}^n$. For $n = 0$, $\mathbb{C}^0$ is defined to be $\{0\}$.

We will use $y = (y^1, \ldots, y^n)$ to denote a point in $\mathbb{C}^n$, and $x = (x^0, \ldots, x^n)$ to denote a point in $\mathbb{C}^{n+1}$.

Let f be a function defined on $\mathbb{C}^{n+1}$. We will often use $f(x)$ to denote the function f, rather than the value of f at the point x. Although formally incorrect, this will simplify the notation and should cause no confusion. Similarly, $g(y)$ may denote a function g on $\mathbb{C}^n$. Thus, we can represent the restriction of $f(x)$ to $\mathbb{C}^n$ by $f(0, y)$. We will use $\equiv$ to denote the equality of two functions.

We will identify $\mathbb{C}^{n+1}$ with its dual space in the usual way. We let $p = (p_0, \ldots, p_n)$ denote an element of this dual space. (Thus, p represents the linear function $(x^0, \ldots, x^n) \to \Sigma\, p_i x^i$.) Thus, the cotangent bundle $T_*(\mathbb{C}^{n+1})$ over $\mathbb{C}^{n+1}$ is the set of all elements $(x; p)$ with $x, p \in \mathbb{C}^{n+1}$. The canonical projection $\pi : T_*(\mathbb{C}^{n+1}) \to \mathbb{C}^{n+1}$ is just the mapping $(x; p) \to x$.

We let D_i denote the operator $\partial/\partial x^i$, and let $D = (D_0, \ldots, D_n)$. Thus we have

$$Df(x) = \left(\frac{\partial f}{\partial x^0}(x), \ldots, \frac{\partial f}{\partial x^n}(x)\right).$$

Note that if f is analytic on an open subset $\mathcal{O}$ of $\mathbb{C}^{n+1}$, then Df maps $\mathcal{O}$ into the dual space of $\mathbb{C}^{n+1}$.

The set of all functions analytic on an open set $\mathcal{O}$ will be denoted by $\mathcal{A}(\mathcal{O})$.

A <u>complex path</u> in $\mathbb{C}^{n+1}$ is a mapping $\gamma : \mathcal{O} \to \mathbb{C}^{n+1}$ on some connected open subset $\mathcal{O}$ of $\mathbb{C}$.

A <u>hypersurface</u> in $\mathbb{C}^{n+1}$ is a subset S such that for every point $x_0 \in S$ there is a neighborhood N of x_0 and a function $f \in \mathcal{A}(N)$ such that

$$S \cap N = \{x \in N : f(x) = 0\}.$$

The point x_0 is a <u>regular</u> point of S if the function f can be chosen such that $Df(x_0) \neq 0$. In this case, N may be chosen so that $S \cap N$ is a submanifold of $\mathbb{C}^{n+1}$.

Let x_0 be a regular point of a hypersurface S. A covector $(x_0; p) \in T_*(\mathbb{C}^{n+1})$ is said to be <u>normal</u> to S if for every vector

$(z^0, \ldots, z^n)$ tangent to S at x^0, $\Sigma\, p_i z^i = 0$. If $S = \{x : f(x) = 0\}$ and $Df(x_0) \neq 0$, then $(x_0; p)$ is normal to S if and only if $p = \lambda\, Df(x_0)$ for some $\lambda \in \mathbb{C}$. Thus, the covectors normal to S at x_0 form a one-dimensional fiber of $T_*(\mathbb{C}^{n-1})$.

Two hypersurfaces are tangent at a point x_0 if there is a covector $(x_0; p)$, $p \neq 0$, which is normal to both of them.

We let δ^i_j denote the usual Kronecker delta. We also define δ^0 by

$$\delta^0 = (\delta^0_0, \delta^0_1, \ldots, \delta^0_n) = (1, 0, \ldots, 0).$$

Observe that $(0, y; \delta^0)$ is normal to $\mathbb{C}^n$, for all $y \in \mathbb{C}^n$.

An analytic function $h(x; p)$ on a subset $\mathcal{N}$ of $T_*(\mathbb{C}^{n+1})$ is said to be homogeneous of degree k in p if $h(x; \lambda p) = \lambda^k h(x; p)$ for all $(x; p) \in \mathcal{N}$ and all non-zero $\lambda \in \mathbb{C}$. We say that h is a polynomial in p of degree m if we can write

$$h(x; p) = \sum_{i_0 + \ldots + i_n \leq m} h_{i_0 \ldots i_n}(x) p_0^{i_0} \ldots p_n^{i_n}$$

for all $(x; p) \in \mathcal{N}$, where the $h_{i_0 \ldots i_n}$ are analytic on a subset $\mathcal{O}$ of $\mathbb{C}^{n+1}$ such that $\mathcal{N} \subset \mathcal{O} \times \mathbb{C}^{n+1}$.

2. Characteristics

We now make some definitions and state some elementary results from the theory of characteristics. An exposition of the theory may be found in [2].

Assume now that $h(x; p)$ is an analytic function on a subset of $T_*(\mathbb{C}^{n+1})$ which is homogeneous in p. A covector $(x_0; p)$ is said to be a characteristic covector of h if $h(x_0; p) = 0$. A regular point x_0 of a hypersurface K is a characteristic point of K for h if some (and thus any) nonzero covector normal to K at x_0 is a characteristic covector of h. The hypersurface K is said to be a characteristic surface of h if each of its points is a characteristic point of K for h. When there is no ambiguity, we will omit the reference to h.

An analytic function $\varphi(x)$ is a characteristic function of h if $h(x; D\varphi(x)) = 0$, and $D\varphi(x) \neq 0$ for all x in the domain of φ. This implies that the hypersurfaces $\{x : \varphi(x) = \text{constant}\}$ are characteristic surfaces of h.

The bicharacteristic equations of h are the Hamiltonian system

(2-1)
$$\frac{dx^i}{dt} \equiv \frac{\partial h}{\partial p_i}(x; p)$$

$$\frac{dp_i}{dt} \equiv -\frac{\partial h}{\partial x^i}(x; p), \qquad i = 0, \ldots, n$$

of complex ordinary differential equations in $T_*(C^{n+1})$. That is, a complex path $t \to (\xi(t); \pi(t))$ satisfies the system (2-1) if and only if

(2-2)
$$\frac{d\xi^i}{dt}(t) \equiv \frac{\partial h}{\partial p_i}(\xi(t); \pi(t))$$

$$\frac{d\pi_i}{dt}(t) \equiv -\frac{\partial h}{\partial x^i}(\xi(t); \pi(t)),$$

where $(\xi(t); \pi(t)) = (\xi^0(t), \ldots, \xi^n(t); \ \pi_0(t), \ldots, \pi_n(t))$.

If, in addition, the path satisfies the condition

(2-3)
$$h(\xi(t); \pi(t)) \equiv 0 ,$$

Then it is called a <u>bicharacteristic strip</u> of h. A path in $\mathbb{C}^{n+1}$ which is the image of a bicharacteristic strip under the canonical projection $T_*(\mathbb{C}^{n+1}) \to \mathbb{C}^{n+1}$ -- i.e., the path $t \to \xi(t)$ -- is called a <u>bicharacteristic curve</u> of h.

If the path $t \to (\xi(t); \pi(t))$ satisfies the bicharacteristic equations of h, then $h(\xi(t), \pi(t))$ is a constant. Hence, condition (2-3) is satisfies if and only if $h(\xi(0); \pi(0)) = 0$. Bicharacteristic strips are therefore obtained by solving the system (2-1) with appropriate initial conditions, using the usual existence and uniqueness theorem of ordinary differential equations.

Suppose $h(x_0; p) = 0$, and let $t \to \xi(t)$ be the bicharacteristic curve obtained by solving (2-2) with initial conditions

$$\xi(0) = x_0, \qquad \pi(0) = p \neq 0.$$

If p is replaced by λp, for $\lambda \in \mathbb{C}$, then the bicharacteristic curve becomes $t \to \xi(\lambda^k t)$, where k is the degree of homogeneity of $h(x; p)$. Therefore, the locus of the bicharacteristic curve is determined by the one-dimensional fiber lying over x_0 in $T_*(\mathbb{C}^{n+1})$ which is spanned by $(x_0; p)$.

Let x_0 be a characteristic point of a hypersurface S for h. Choose any $p \neq 0$ such that $(x_0; p)$ is normal to S. Then $h(x_0; p) = 0$. The locus of the bicharacteristic curve determined by $(x_0; p)$ is independent of the choice of p. The union of these loci for all characteristic points of S is called the characteristic set tangent to S. If this set is a submanifold, then it is a characteristic surface of h, which is tangent to S at each characteristic point of S.

Now assume that $h(x; p)$ is a homogeneous polynomial in p of degree m. Suppose that φ is a characteristic function of h such that $\varphi(0, y^1, \ldots, y^n) \equiv y^1$. Then

$$h(0, y; D\varphi(0, y)) \equiv h(0, y; D_0\varphi(0, y), 1, 0, \ldots, 0) \equiv 0.$$

In other words, for each y, $\tau = D_0\varphi(0, y)$ is a root of the polynomial equation

$$h(0, y; \tau, 1, 0, \ldots, 0) = 0 \tag{2-4}$$

Conversely, assume that 0 is not a characteristic point of $\mathbb{C}^n$ for h. Then $h(0, y; \delta^0)$, which is the coefficient of τ^m in (2-4), is non-zero for all y in some neighborhood N of 0 in $\mathbb{C}^n$. Therefore, (2-4) is a polynomial equation of degree m in τ, for all $y \in N$.

Assume that there is a function $\alpha \in \mathcal{A}(N)$ such that $\tau = \alpha(y)$ is a root of (2-4) for all $y \in N$. Moreover, assume that $\alpha(y)$ is not a multiple root, so $\partial h/\partial p_0(0, y; \alpha(y), 1, 0, \ldots, 0) \neq 0$ for all $y \in N$. In this case, we will show that a characteristic function φ of h can be found on a neighborhood of N in $\mathbb{C}^{n+1}$ satisfying

$$\varphi(0, y) \equiv y^1, \quad D_0\varphi(0, y) \equiv \alpha(y) \tag{2-5}$$

on N.

3. Singularities

We now state some definitions and results from the theory of functions of several complex variables. Proofs of the results can be found in [1].

Let U be a connected open subset of $\mathbb{C}^{n+1}$, $\varphi \in \mathcal{O}(U)$ and $S = \{x \in U : \varphi(x) = 0\}$. We first state:

Proposition 3.1: Let $x_0 \in S$ with $D\varphi(x_0) \neq 0$. If $g \in \mathcal{O}(U-S)$, then there is a neighborhood N of x_0 in U, a neighborhood M of 0 in $\mathbb{C}$ and a function $h(t, x) \in \mathcal{O}((M-\{0\}) \times N)$ such that $g(x) \equiv h(\varphi(x), x)$ on $N-S$.

Definition 3.2: Let $g \in \mathcal{O}(U-S)$. Then g is said to have a pole, or polar singularity on S if $[\varphi(x)]^k g(x)$ is analytic on U, for some integer k. Otherwise, g is said to have an essential singularity on S.

Definition 3.3: A q-valued analytic mapping $g : U \to \mathbb{C}^m$ is a multiple valued mapping such that for every simply-connected open subset N of U, the restriction of g to N consists of q branches $g_i : N \to C^m$ such that:

(1) g_i is analytic on N

(2) g_i can be obtained from any g_j by analytic continuation along a path in U

(3) analytic continuation of g_i q times around any closed path in U yields the same mapping g_i

The set of all q-valued analytic (complex-valued) functions on U is denoted by $\mathcal{A}_q(U)$.

(We could more elegantly defing g to be an analytic mapping on the simply-connected covering space of U.)

Note that the q branches g_i need not be distinct. However, it is easy to show that the number of distinct branches must divide q. Of course, a 1-valued analytic mapping is just an ordinary analytic mapping.

If f and $g \in \mathcal{A}_q(U)$, then $f + g \in \mathcal{A}_q(U)$ is defined by specifying one branch of it on any open subset of U. This can be done by specifying a branch of f and a branch of g on a neighborhood of any point of U.

Proposition 3.4: Let $g \in \mathcal{A}_q(U-S)$. Then the following two conditions are equivalent:

(1) For any $x_0 \in S$, there is a neighborhood N of x_0 such that g is bounded on $N-S$.

(2) For any $x_0 \in S$, there is a neighborhood N of x_0 and a function $h(t, x) \in \mathcal{A}(\mathbb{C} \times N)$ such that $h(t, x)$ is a monic polynomial in t of degree q satisfying $h(g(x), x) \equiv 0$ on $N-S$.

If g satisfies these conditions, it is said to be an algebraic function on U with singularities on S. An algebraic mapping on U is a q-valued analytic mapping on U-S whose coordinate functions are algebraic functions.

Observe that a 1-valued algebraic mapping on U is analytic on U.

In analogy to Definition 3.2, we make:

Definition 3.5: A function $g \in \mathcal{A}_q(U-S)$ is said to have a pole, or polar singularity on S if $[\varphi(x)]^k g(x)$ is an algebraic function on U for some integer k. Otherwise, g is said to have an essential singularity on S.

The analogue of Proposition 3.1 is:

Proposition 3.6: Let $x_0 \in S$ with $D\varphi(x_0) \neq 0$. If $g \in \mathcal{A}_q(U-S)$, then there is a neighborhood N of x_0 in U, a neighborhood M of 0 in C and a function $h \in \mathcal{A}([M-\{0\}] \times N)$ such that $g(x) \equiv h([\varphi(x)]^{1/q}, x)$ on N-S. If g is algebraic, then we can choose $h \in \mathcal{A}(M \times N)$.

4. The Cauchy Problem

Let U be an open subset of $\mathbb{C}^{n+1}$. An operator on U is defined to be an analytic linear differential operator -- that is, a mapping $x \to a(x; D)$ such that $a(x; p)$ is an analytic function on $U \times C^{n+1}$ which is a polynomial in p. The principal part of the operator a(x; D) is the sum of the highest order terms in $p_0, \ldots, p_n$ of a(x; p). The principal part of a(x; D) is thus a homogeneous polynomial in p of degree equal to the order of the operator a. The characteristic surfaces, functions, etc. of a(x; D) are defined to be the characteristic surfaces, functions, etc. of its principal part.

We will be considering solution functions u(x) of the following Cauchy problem:

$$(4\text{-}1) \qquad a(x; D)u(x) \equiv v(x)$$

$$(D_0)^j u(0, y) \equiv w^j(y), \qquad j = 0, \ldots, m-1;$$

where a(x; D) is an operator of order m on a neighborhood of 0, v(x) a function on a neighborhood of 0 in $\mathbb{C}^{n+1}$, and the $w^j(y)$ functions on a neighborhood of 0 in $\mathbb{C}^n$. We will seek a solution function u(x) defined on a neighborhood of 0 in $\mathbb{C}^{n+1}$.

Note that any local results for the Cauchy problem (4-1) can be applied to the Cauchy problem on an arbitrary finite-dimensional analytic manifold, with initial data on a submanifold of codimension one.

The classical Cauchy-Kowalewski Theorem states that if:

(1) v(x) and the $w^j(y)$ are analytic at 0, and

(2) 0 is not a characteristic point of $\mathbb{C}^n$ for a; then there

exists a unique solution function $u(x)$ which is analytic on a neighborhood of 0. (See [9] for a more precise statement of the theorem.)

Hamada [5] considered the case in which the w^j have singularities along a submanifold of $\mathbb{C}^n$ containing 0. Without loss of generality, this submanifold can be taken to be $\mathbb{C}^{n-1}$. Let $h(x;p)$ be the principal part of $a(x;D)$. Under the hypothesis that the m roots of (2-4) are distinct for all y in some neighborhood of 0, he proved the existence and uniqueness of a multiple valued analytic solution $u(x)$ having singularities along the characteristic surfaces $\{x : \varphi(x) = 0\}$ emanating from $\mathbb{C}^{n-1}$, where the functions $\varphi(x)$ are the ones defined by (2-5) for the different roots $\alpha(y)$ of (2-4).

We will generalize Hamada's result in the following ways:

(1) We will allow certain cases of multiple roots of (2-4).

(2) $v(x)$ may have a singularity along a certain type of submanifold of $\mathbb{C}^{n+1}$ containing 0.

(3) $v(x)$ and the $w^j(y)$ may be q-valued functions.*

Leray [6] and Garding, Kotake and Leray [4] considered the case in which 0 is a characteristic point of $\mathbb{C}^n$. They showed that the solution function $u(x)$ can be <u>uniformized</u>. That is, there exist an analytic function $\bar{u}(x)$ on a subset of $\mathbb{C}^{n+1}$ and an analytic

*Wagshal [10] has recently also covered this case, in the course of extending Hamada's results to systems of equations.

mapping σ from a subset of $\mathbb{C}^{n+1}$ into $\mathbb{C}^{n+1}$ such that $\bar{u} = u \circ \sigma$. Under certain conditions, σ^{-1} is an algebraic mapping, so $u = \bar{u} \circ \sigma^{-1}$ is an algebraic function. In this case, the singularities of u are contained in the characteristic set tangent to $\mathbb{C}^n$.

We will generalize their results to allow the $w^j(y)$ to be q-values analytic functions with singularities along $\mathbb{C}^{n-1}$.

The above results describe how the singularities of the solution are propagated from the singularities of the initial date. For equations in two dimensions $(n = 1)$, we will give a complementary result describing the propagation of analyticity. More precisely, assuming certain restrictions on $a(x; D)$, for a simply-connected domain of analyticity of the initial date we will describe a domain of analyticity of the solution. This generalizes a known result for real hyperbolic operators. (See [8].)

5. Operators of Constant Multiplicity

Definition 5.1: Let $h(x;p)$ be a homogeneous polynomial in p, analytic for x in a neighborhood of 0 in $\mathbb{C}^{n+1}$, such that $h(0; \delta^0) \neq 0$. Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, $\xi \neq 0$. Then h is said to be of constant multiplicity at 0 in the direction of ξ if there exists a neighborhood N of 0 in $\mathbb{C}^{n+1}$, and $h_1(x; p), \ldots, h_s(x; p) \in \mathcal{O}(N \times \mathbb{C}^{n+1})$ which are homogeneous polynomials in p, such that:

(1) $h(x; p) \equiv [h_1(x; p)]^{k_1} \ldots [h_s(x; p)]^{k_s}$ on N

(2) Let m_i be the degree of $h_i(x;p)$, and let $\tau = \alpha_i^{(j)}$, $j = 1, \ldots, m_i$, be the roots of the polynomial $h_i(0; \tau, \xi_1, \ldots, \xi_n)$. Then the $\Sigma\, m_i$ numbers $\alpha_i^{(j)}$ are distinct.

If $s = 1$ and $k_1 = 1$, then $h(x;p)$ is said to have multiplicity one at 0 in the direction of ξ.

Since the $\alpha_i^{(j)}$ are distinct, they can be extended to functions analytic on a neighborhood of 0 in $\mathbb{C}^n$ such that $\tau = \alpha_i^{(j)}(y)$ are the roots of $h_i(y; \tau, \xi_1, \ldots, \xi_n)$.

Note that multiplying ξ by a scalar λ just causes the $\alpha_i^{(j)}$ to be multiplied by λ. We can therefore make the following:

Definition 5.2: Let $a(x; D)$ be an operator on a neighborhood of 0 such that 0 is not a characteristic point of $\mathbb{C}^n$. Let S be a submanifold of $\mathbb{C}^n$ of co-dimension one containing 0, and let $(0; \xi) \in T_*(\mathbb{C}^n)$ be normal to S, $\xi \neq 0$. Then $a(x; D)$ is called an operator of constant multiplicity (multiplicity one) at 0 in the direction of S if its principal part is of constant multiplicity (multiplicity one) in the direction of ξ.

The following Theorem of Matsuura [7] is of interest in connection with these definitions. However, we will not use it.

<u>Theorem</u>: The following conditions are equivalent:

(1) There are constants ν_i and analytic functions $\tau_i(x; \xi)$ on $N \times (\mathbb{R}^n - \{0\})$ for some neighborhood N of 0 in $\mathbb{C}^{n+1}$, such that

$$h(x; \tau, \xi) \equiv [\tau - \tau_1(x; \xi)]^{\nu_1} \dots [\tau - \tau_r(x; \xi)]^{\nu_r} .$$

(2) $$h(x; p) \equiv [h_1(x; p)]^{k_1} \dots [h_s(x; p)]^{k_s} ,$$

where the $h_i(x; p)$ satisfy the conditions of Definition 5.1 for all $\xi \in \mathbb{R}^n - \{0\}$.

Of course, (2) implies that h has constant multiplicity in the direction of ξ for all $\xi \in \mathbb{R}^n - \{0\}$.

(<u>Note</u>: Matsuura actually states a somewhat different result. E.g., he only considers real hyperbolic operators -- for which the $\tau_i(x; \xi)$ are real when $x \in \mathbb{R}^n$. However, the proof of the above theorem is identical to the proof given in [7].)

6. Extension of Hamada's Result

Let $a(x; D)$ be an operator of order m on a neighborhood $\mathcal{O}$ of 0, which is of constant multiplicity at 0 in the direction of $\mathbb{C}^{n-1}$. Let $h(x; p)$ be its principal part, and let $\alpha^{(1)}, \ldots, \alpha^{(r)}$ be analytic functions on a neighborhood of 0 in $\mathbb{C}^n$ such that $\tau = \alpha^{(i)}(y)$ are the distinct roots of $h(0, y; \tau, 1, 0, \ldots, 0)$.

Let $\varphi^{(i)}(x)$ be the characteristic functions of $h(x; p)$ such that

$$\varphi^{(i)}(0, y^1, \ldots, y^n) \equiv y^1, \quad D_0\varphi^{(i)}(0, y) \equiv \alpha^{(i)}(y),$$

and let $K^{(i)} = \{x : \varphi^{(i)}(x) = 0\}$. Then the $K^{(i)}$ are distinct characteristic surfaces of $a(x; D)$ passing through 0, no two of which are tangent at 0. Moreover, $K^{(i)} \cap \mathbb{C}^n \subset \mathbb{C}^{n-1}$.

Let $K^{(0)}$ be a submanifold of $\mathbb{C}^{n+1}$ of co-dimension one containing 0 which is not tangent to $\mathbb{C}^n$, such that $K^{(0)} \cap \mathbb{C}^n \subset \mathbb{C}^{n-1}$. Assume that either $K^{(0)} = K^{(i)}$ for some i, or else $K^{(0)}$ is not tangent to any other $K^{(i)}$ at 0. Let $\varphi^{(0)}(x) \in \mathcal{A}(N)$ for some neighborhood N of 0 be such that

$$\varphi^{(0)}(0, y^1, \ldots, y^n) \equiv y^1$$

and

$$K^{(0)} \cap N = \{x : \varphi(x) = 0\}.$$

If $K^{(0)} = K^{(i)}$ for $i \geq 1$; take $\varphi^{(0)} = \varphi^{(i)}$.

We can now state our result:

Theorem I: Let $a(x; D)$, $\mathcal{O}$, $\varphi^{(i)}$ and $K^{(i)}$ be above, let $w^j(y) \in \mathcal{A}_q(\mathcal{O} \cap \mathbb{C}^n - \mathbb{C}^{n-1})$ for $j = 0, \ldots, m-1$ and let $v(x) \in \mathcal{A}_q(\mathcal{O} - K^{(0)})$. Then

1. (a) There exists a unique solution function $u(x)$ of the Cauchy problem (4-1), defined on a neighborhood U of 0, of the form

$$u(x) \equiv \sum_{i=0}^{r} F^{(i)}(x) + G^{(i)}(x) \log[\varphi^{(i)}(x)], \tag{6-1}$$

where $F^{(i)} \in \mathcal{A}_q(U - K^{(i)})$ and $G^{(i)} \in \mathcal{A}(U)$.

(b) If $a(x; D)$ has multiplicity one, and the $w^j(y)$ and $v(x)$ have polar singularities, then each $F^{(i)}(x)$ has a polar singularity along $K^{(i)}$.

2. (a) If the $w^j(y)$ and $v(x)$ are algebraic functions, then each $G^{(i)}(x) \equiv 0$.

(b) If, in addition, $a(x; D)$ has multiplicity one, then each $F^{(i)}$ is an algebraic function on U with singularity on $K^{(i)}$.

Note: For multiple valued functions $w^j(y)$, $v(x)$, finding a solution function $u(x)$ of the Cauchy problem (4-1) has the following meaning. Let N be a simply-connected open subset of $\mathcal{O} - \bigcup_{i=0}^{r} K^{(i)}$ having 0 as a boundary point. Choose a branch of $v(x)$ on N and a branch of each $w^j(y)$ on $N \cap \mathbb{C}^n$. Then $u(x)$ can be chosen so that it has a branch on $N \cap U$ which satisfies (4-1) for the chosen branches of v and w^j.

Analytic continuation along a path lying in $\mathbb{C}^n - \mathbb{C}^{n-1}$ clearly yields another branch of $u(x)$ which is a solution of (4-1) for other branches of w^j and v. At most q distinct branches are obtained in this way, corresponding to the q branches of w^j and v. However, continuation along a path going outside $\mathbb{C}^n$ may yield a branch of u which does not satisfy the initial conditions $(D_0)^j u(0, y) \equiv w^j(y)$.

Remark 6.1: In part 1 of the theorem, the conclusions are not changed if we add to each $w^j(y)$ a function $g^j(y)\log(y^1)$, and add to $v(x)$ a function $h(x)\log[\varphi^{(0)}(x)]$, where $g^j \in \mathcal{A}(\mathcal{O} \cap \mathbb{C}^n)$ and $h \in \mathcal{A}(\mathcal{O})$.

Hamada proved part 1 for $q = 1$, v analytic on $\mathcal{O}$ and $a(x; D)$ of multiplicity one. The conclusion of part 1 (b) is false without the assumption of multiplicity one. To see this, consider the following Cauchy problem on $\mathbb{C}^2$:

$$\frac{\partial^2 u}{\partial t^2}(t, y) - \frac{\partial u}{\partial y}(z, y) \equiv 1/y.$$

$$u(0, y) \equiv \frac{\partial u}{\partial t}(0, y) \equiv 0.$$

Then $h(t, y; p_0, p_1) \equiv (p_0)^2$, so $r = 1$, $\alpha^{(1)}(y) \equiv 0$ and $\varphi^{(1)}(t, y) \equiv y$. Term by term differentiation shows that we have the solution

$$u(t, y) \equiv \sum_{k=0}^{\infty} (-1)^k \frac{k!}{[2(k+1)]!} \frac{t^{2(k+1)}}{y^{k+1}} .$$

Since the sum is absolutely convergent on $\{(t, y) \in \mathbb{C}^2 : y \neq 0\} = \mathbb{C}^2 - K^{(1)}$, u is analytic on this set. Obviously, u has an essential singularity on $K^{(0)}$.

Similarly, the solution

$$u(t, y) \equiv t^2 y^{1/2} + \sum_{k=1}^{\infty} (1/2) \frac{(-1/2)\ldots(1/2-(k-1))}{[2(k+1)]!} \frac{t^{2(k+1)}}{y^{k-1/2}}$$

of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2}(t, y) - \frac{\partial u}{\partial y}(t, y) \equiv y^{1/2}$$

$$u(0, y) \equiv \frac{\partial u}{\partial t}(0, y) \equiv 0$$

shows that hypothesis of multiplicity one is necessary in part 2 (b).

7. The Cauchy Problem at a Characteristic Point

We now consider the Cauchy problem (4-1) when 0 is a characteristic point of $\mathbb{C}^n$. Let $a(x; D)$ be an operator of order m on an open neighborhood $\mathcal{O}$ of 0 with principal part $h(x; p)$, and let $g(x; p) \equiv h(x; p)/(p_0)^{m-1}$. Then $g(x; p)$ is analytic on a neighborhood of $(0, \delta^0)$ in $T_*(\mathbb{C}^{n+1})$, and is homogeneous of degree 1 in p. Moreover, $g(x; \delta^0)$ equals $h(x; \delta^0)$, the coefficient of $(D_0)^m$ in a.

Define a mapping $\sigma : N \to \mathbb{C}^{n+1}$ for some neighborhood N of 0 in $\mathbb{C}^{n+1}$ by letting $t \to (\sigma(t, y); \pi(t, y))$ be the solution path of the bicharacteristic equations of g:

(7-1)
$$\frac{d\sigma^i}{dt}(t, y) \equiv \frac{\partial g}{\partial p_i}(\sigma(t, y); \pi(t, y))$$

$$\frac{d\pi_i}{dt}(t, y) \equiv -\frac{\partial g}{\partial x^i}(\sigma(t, y); \pi(t, y))$$

with initial conditions

(7-2)
$$\sigma(0, y; \delta^0) = y$$

$$\pi(0, y; \delta^0) = \delta^0$$

for $y \in \mathbb{C}^n$. The analyticity of σ (and π) follows from the theory of ordinary differential equations. (See [3].) Note that σ is the identity mapping on $\mathbb{C}^n$.

Let $T = \{(0, y) \in \mathbb{C}^n : g(0, y; \delta^0) = 0\}$, so T is the set of characteristic points of $\mathbb{C}^n$. We will prove:

<u>Proposition 7.1</u>: Assume $\mathbb{C}^n$ is not a characteristic surface of $a(x; D)$. Then σ^{-1} is an analytic mapping on a neighborhood of $\mathcal{O} \cap \mathbb{C}^n - T$ in $\mathbb{C}^{n+1}$.

Now let $K = \{\sigma(t, y) : y \in T\}$. Then K is the characteristic set of $a(x; D)$ tangent to $\mathbb{C}^n$.

Garding, Kotake and Leray defined a characteristic point $y \in T$ to be exceptional if the bicharacteristic curve $t \to \sigma(t, y)$ lies entirely in $\mathbb{C}^n$. They then proved the following result:

<u>Theorem</u>: 1. If 0 is a non-exceptional characteristic point of $\mathbb{C}^n$ for $a(x; D)$, then σ^{-1} is an algebraic mapping on a neighborhood of 0, with its singularities contained in K.

2. Assume that in some neighborhood of 0 in $\mathbb{C}^n$:

(a) T is a submanifold of $\mathbb{C}^n$,

(b) The function $y \to g(0, y; \delta^0)$ has a zero of order $q-1$ at each point of T,

(c) The vector $(\partial g/\partial p_1(0, y; \delta^0), \ldots, \partial g/\partial p_n(0, y; \delta^0))$ is not tangent to T for any $y \in T$.

Then a neighborhood of 0 in $\mathbb{C}^{n+1}$, K is a submanifold of $\mathbb{C}^{n+1}$ (of co-dimension one) and σ^{-1} is a q-valued algebraic mapping with singularities contained in K.

Now assume that 0 is a characteristic point of $\mathbb{C}^n$, but $\mathbb{C}^n$ is not a characteristic surface of $a(x; D)$. In other words, $g(0; \delta^0) = 0$, but $g(0, y; \delta^0) \not\equiv 0$. Since $h(x; p)$ is an mth degree polynomial in p_0 with leading coefficient $g(x; \delta^0)$, we can define a homogeneous polynomial $\tilde{h}(x; p)$ in p of order m by setting

$$\tilde{h}(x; g(x; \delta^0)p_0, p_1, \ldots, p_n) \equiv [g(x; \delta^0)]^{m-1} h(x; p). \tag{7-3}$$

Then $\tilde{h}(x; p)$ is analytic on $\mathcal{O} \times \mathbb{C}^{n+1}$. Moreover, $h(x; \delta^0) \equiv 1$, so 0 is not a characteristic point of $\mathbb{C}^n$ for $\tilde{h}$.

We now state our main results.

Theorem II: Let $a(x; D)$, $\mathcal{O}$, σ and $\tilde{h}(x; p)$ be as above. Assume that $\tilde{h}$ has constant multiplicity in the direction of $\mathbb{C}^{n-1}$. Then there exist analytic functions $\bar{\varphi}^{(1)}(x), \ldots, \bar{\varphi}^{(r)}(x)$ on a neighborhood of 0 in $\mathbb{C}^{n+1}$ with $\bar{\varphi}^{(i)}(0, y^1, \ldots, y^n) \equiv y^1$, $D\bar{\varphi}^{(i)}(0, y) \neq 0$ for all y, and submanifolds $\bar{K}^{(i)} = \{x : \bar{\varphi}^{(i)} = 0\}$ such that:

1. (a) If $w^j(y) \in \mathcal{A}_q(\mathcal{O} \cap \mathbb{C}^n - \mathbb{C}^{n-1})$ for $j = 0, \ldots, m-1$, and $v(x) \in \mathcal{A}(\mathcal{O})$, then the Cauchy problem (4-1) has a

unique solution of the form $u(x) \equiv \bar{u}(x) \circ \sigma^{-1}$, where

$$\bar{u}(x) \equiv \sum_{i=1}^{r} \bar{F}^{(i)}(x) + \bar{G}^{(i)}(x)\log[\bar{\varphi}^{(i)}(x)]$$

with $\bar{F}^{(i)} \in \mathcal{A}_q(U-\bar{K}^{(i)})$ and $\bar{G}^{(i)} \in \mathcal{A}(U)$ for some neighborhood U of 0.

(b) If $\tilde{h}(x;p)$ has multiplicity one, and the w^j have polar singularities on $\mathbb{C}^{n-1}$, then each $\bar{F}^{(i)}$ has a polar singularity on $\bar{K}^{(i)}$.

2. If the w^j are algebraic functions, then each $\bar{G}^{(i)}(x) \equiv 0$. If, in addition, $\tilde{h}$ has multiplicity one, then each $\bar{F}^{(i)}$ is algebraic on U with singularities on $\bar{K}^{(i)}$.

If the $w^j(y)$ have polar singularities along the set T of characteristic points of $\mathbb{C}^n$, the following theorem may apply.

Theorem III: Let $a(x;D)$, $h(x;p)$, $\mathcal{O}$ and σ be as above. If $[h(0,y;\delta^0)]^j w^j(y) \in \mathcal{A}(\mathcal{O})$ for $j = 0,\ldots,m-1$, then the Cauchy problem (4-1) has a unique solution of the form $u(x) \equiv \bar{u}(x) \circ \sigma^{-1}$, where $\bar{u}(x)$ is analytic on a neighborhood of 0.

By Proposition 7.1, the solution functions $u(x)$ of Theorems II and III are defined on a neighborhood of $\mathcal{O} \cap \mathbb{C}^n - T$ in $\mathbb{C}^{n+1}$. The note following Theorem I in Section 5 applies easily in this case to define precisely what is meant by a multiple-valued solution of the Cauchy problem.

Remark 7.2: As in Theorem I, the conclusions of part 2 of Theorem II is not changed if a function $g^j(y) \log(h^1)$ is added to $w^j(y)$, where $g^j \in \mathcal{A}(\mathcal{O} \cap \mathbb{C}^n)$.

Note that if 0 is a non-exceptional point, the Theorem of Garding, Kotake and Leray implies that the solution function $u(x)$ of Theorem III is an algebraic function.

8. Analyticity of the Solution in Two Dimensions

We first make some preliminary definitions.

Definition 8.1: A star-shaped domain in $\mathbb{C}$ is an open neighborhood N of 0 such that if $t \in N$, then $\rho t \in N$ for all $\rho \in \mathbb{R}$ such that $0 \leq \rho \leq 1$.

For the following definition, we consider $\mathbb{C}$ to have the differentiable structure of a two-dimensional vector space over $\mathbb{R}$.

Definition 8.2: A star-like domain in $\mathbb{C}$ is an open neighborhood $\mathcal{D}$ of 0 together with a bijective mapping $\eta : N \to \mathcal{D}$ on a star-shaper domain N such that:

(1) $\eta(0) = 0$

(2) η and η^{-1} are continuously differentiable.

Note that a star-shaped domain is also star-like, where η may be taken to be the identity mapping.

Assume that we have chosen a star-like domain $\mathcal{D}$. Let $n = 1$, and let $a(x; D)$ be an operator of order m on an open set $\mathcal{O} \subset \mathcal{D} \times \mathbb{C}$, with principal part $h(x; p)$. We assume that there are functions $\alpha^{(i)} \in \mathcal{A}(\mathcal{O})$ such that

(8-1) $$h(x;p) \equiv (p_0-\alpha^{(1)}(x)p_1)\ \ldots\ (p_0-\alpha^{(m)}(x)p_1).$$

(For any operator on $\mathbb{C}^2$ of order m whose coefficient of $(D_0)^m$ is 1, for almost all points x_0 in its domain we can choose a neighborhood $\mathcal{O}$ of x_0 and $\alpha^{(i)} \in \mathcal{A}(\mathcal{O})$ satisfying (8-1).)

This assumption about $a(x; D)$ implies that for any $x_0 \in \mathcal{O}$, we can find m characteristic surfaces (not necessarily distinct) passing through x_0, as follows. Let $x_0 = (t_0, y_0)$ and define the complex path $\gamma_{x_0}^{(i)}$ in $\mathcal{O}$ to be the solution of

(8-2) $$\frac{d\gamma_{x_0}^{(i)}}{dt}(t) \equiv (1, -\alpha^{(i)}[\gamma_{x_0}^{(i)}(t)])$$

$$\gamma_{x_0}^{(i)}(t_0) = x_0.$$

The theory of ordinary differential equations guarantees the analyticity of $\gamma_{x_0}^{(i)}$ on a neighborhood of t_0 in $\mathbb{C}$. The locus of the path $\gamma_{x_0}^{(i)}$ is a characteristic surface for $a(x; D)$, since if $\gamma_{x_0}^{(i)}(t) = x$, then $(x; \alpha^{(i)}(x), 1) \in T_*(\mathbb{C}^2)$ is tangent to this surface.

Note that if x_1 lies on the curve $\gamma_{x_0}^{(i)}$, then $\gamma_{x_0}^{(i)} \equiv \gamma_{x_1}^{(i)}$. Also note that t is the first coordinate of the point $\gamma_{x_0}^{(i)}(t) \in \mathbb{C}^2$.

<u>Definition 8.3</u>: Let η be as in Definition 8.2, $x_0 = (t_0, y_0)$, and

$\gamma_{x_0}^{(i)}$ as above. Let $t_0 = \eta(\rho e^{i\theta})$, where $\rho, \theta \in \mathbb{R}$ and $\rho \geq 0$. The <u>dependence path</u> $\beta_{x_0}^{(i)} : [0, \rho] \to \mathcal{O}$ is defined by

$$\beta_{x_0}^{(i)}(\tau) \equiv \gamma_{x_0}^{(i)}[\eta(\tau e^{i\theta})]$$

if $\eta(\tau e^{i\theta})$ is in the domain of $\gamma_{x_0}^{(i)}$ for all $\tau \in [0, \rho]$. Otherwise, $\beta_{x_0}^{(i)}$ is undefined.

Note that $\beta_{x_0}^{(i)}(\rho) = x_0$ and $\beta_{x_0}^{(i)}(0) \in \mathbb{C}$. In fact, $\beta_{x_0}^{(i)}(0)$ is the intersection of the locus of $\gamma_{x_0}^{(i)}$ with $\mathbb{C}$. (Recall that $\eta(0) = 0$, and $\gamma_{x_0}^{(i)}(t) = (t, y)$ for some $y \in \mathbb{C}$.) Hence, $\beta_{x_0}^{(i)}$ is a (real) path from the initial plane $\mathbb{C}$ to x_0, lying in a characteristic surface.

<u>Definition 8.4</u>: Let V be an open subset of $\mathcal{O} \cap \mathbb{C}$.

(1) For any compact set $K \subset \mathcal{O}$, $\mathcal{J}(K)$ is defined to be the maximal subset of K having the following property: For any $x \in K$ and each $i = 1, \ldots, m$, the path $\beta_x^{(i)}$ is defined and lies in K.

(2) The influence domain $\mathcal{J}(V)$ is the union of the $\mathcal{J}(K)$, for all compact subsets K of $\mathcal{O}$ with $K \cap \mathbb{C} \subset V$.

Note that if any collection of subsets of K have the property stated in part (1) of the definition, then their union also has that property. Hence, $\mathcal{J}(K)$ is well-defined. Obviously $K \cap \mathbb{C} \subset \mathcal{J}(K)$, so $V \subset \mathcal{J}(V)$. Remember that $\mathcal{J}(V)$ depends upon $a(x; D)$ and $\mathcal{B}$.

We will prove:

Proposition 8.5: (1) If V is an open subset of $\mathcal{O} \cap \mathbb{C}$, then $\mathcal{J}(V)$ is an open subset of $\mathcal{O}$.

(2) If V is simply-connected, then $\mathcal{J}(V)$ is simply connected.

We now consider the Cauchy problem (4-1) for $a(x; D)$. Theorem I shows (under a more restrictive hypothesis on $a(x; D)$ than we are now making) that if $v(x)$ or $w^j(y)$ has a singularity at $y \in \mathbb{C}$, then $u(x)$ has singularities along the curves $\gamma_y^{(i)}$. But x lies on the path $\gamma_y^{(i)}$ if and only if $y = \beta_x^{(i)}(0)$. Hence, we expect that u has a singularity at x only if v or one of the w^j has a singularity at $\beta_x^{(i)}(0)$, or if v has a singularity at x. In other words, if v is analytic at x, and v and the w^j are analytic at $\beta_x^{(i)}(0)$ for each i, then u should be analytic at x. Our actual result is the following:

Theorem IV: Let $a(x; D)$, $\mathcal{O}$ and $\mathcal{B}$ be as above, and let V be a simply-connected open subset of $\mathcal{O} \cap \mathbb{C}$. If $v(x) \in \mathcal{A}(\mathcal{J}(V))$ and $w^j(y) \in \mathcal{A}(V)$ for $j = 0, \ldots, m-1$, then the Cauchy problem (4-1) has a unique solution $u(x) \in \mathcal{A}(\mathcal{J}(V))$.

Now consider $\mathbb{R}^2$ to be a subset of $\mathbb{C}^2$ in the obvious way, and call an element of $\mathbb{C}^2$ real if it lies in $\mathbb{R}^2$. Let $\mathcal{D}$ be a star-shaped domain in $\mathbb{C}$ (so η is the identity mapping). The operator a(x; D) is said to be a real hyperbolic operator if the $\alpha^{(i)}(x) \in \mathbb{R}$ for all real $x \in \mathcal{O}$.

It is clear from (3-2) that the curves $\beta_x^{(i)}$ lie in $\mathbb{R}^2$ for all real x if and only if a(x; D) is a real hyperbolic operator. In this case, the $\beta_x^{(i)}$ are the real characteristic curves passing through x, and the real part of $\mathcal{J}(V)$ is the influence domain of the real part of V, in the usual sense of influence domain for a real hyperbolic operator. (See [9].) Thus, Theorem IV contains the known result for real hyperbolic operators.

Theorem IV also helps explain why this result is only true for a hyperbolic operator. For any other operator, the real part of $\mathcal{J}(V)$ depends upon the non-real part of V. In other words, complex singularities of the initial date can propagate along characteristic curves into $\mathbb{R}^2$.

II. PROOF OF THEOREM I

In Section 9, we show that it suffices to prove Theorem I under certain simplifying assumptions. Using these assumptions, in Section 10 we obtain the solutions of the Cauchy problem in the form of infinite sums of functions. The rest of the chapter is devoted to proving the convergence of these formal series solutions. In Section 13, bounds are obtained for the individual terms of the series, using results from Section 12. With these bounds, convergence is proved in Section 14.

The proof of the existence of the characteristic functions is deferred until Chapter IV, where it is more convenient for it to appear.

9. Reduction of the Problem

We now show that it is sufficient to prove Theorem I with the following additional assumptions.

(1) $w^j(y) \equiv 0$ for each $j = 0, \ldots, m-1$

(2) $\varphi^{(0)} \equiv \varphi^{(1)}$

(3) If $h(x;p)$ is the principal part of $a(x;D)$, then

$$h(x;p) \equiv h_1(x;p) \ldots h_s(x;p),$$

where each $h_i(x;p)$ is a homogeneous polynomial of degree r in p which is analytic on a neighborhood of $\{0\} \times \mathbb{C}^{n+1}$, is of multiplicity one in the direction of $\mathbb{C}^{n+1}$, and has $\varphi^{(1)}, \ldots, \varphi^{(r)}$ as characteristic functions.

Note that $h(x;p)$ is of multiplicity one if and only if $s = 1$.

Since $K^{(0)} = \{x: \varphi^{(0)}(x) = 0\}$ is not tangent to $\mathbb{C}^n$ at 0, we can easily find a neighborhood N of 0 in $\mathbb{C}^{n+1}$ and an analytic mapping $\psi : N \to \mathbb{C}^n$ such that $\psi(0,y) \equiv y$ and $\psi(x) \equiv (\varphi^{(0)}(x), \ldots)$, so $K^{(0)} \cap N = \psi^{-1}(\mathbb{C}^{n-1})$. Let $W^j(x) \equiv w^j \circ \psi(x)$. Then the W^j satisfy the hypotheses for v in all parts of Theorem I.

Now let

$$U(x) \equiv u(x) - \sum_{j=0}^{m-1} (x^0)^j W^j(x)$$

where $x = (x^0, \ldots, x^n)$. Then $u(x)$ is a solution of the Cauchy problem (4-1) if and only if $U(x)$ solves the Cauchy problem

$$\text{(9-1)} \qquad a(x;D)\, U(x) \equiv v(x) + a(x;D) \left[\sum_{j=0}^{m-1} (x^0)^j W^j(x)\right]$$

$$(D_0)^j\, U(0,y) \equiv 0, \; j = 0, \ldots, m-1 \; .$$

If (4-1) satisfies the hypotheses of Part 1 of Theorem I, then (9-1) does also. However, this is not true of Part 2, since the fact that the $W^j(x)$ are algebraic functions does not mean that $a(x;D)\,[\Sigma (x^0)^j W^j(x)]$ is algebraic. Our proof of Part 2 (a) will remain valid when $v(x)$ is a function of the form $b(x;D)\,V(x)$, where $V(x)$ is an algebraic function and $b(x;D)$ is any operator. For Part 2 (b), we will indicate in Section 10 why the solution function $U(x)$ of (9-1) is algebraic when $a(x;D)$ has multiplicity one, even if $a(x;D)\,[\Sigma (x^0)^j W^j(x)]$ is not algebraic. Thus, replacing (4-1) by (9-1) allows us to assume that the $w^j(y) \equiv 0$.

Now renumber the $\varphi^{(i)}$ and $\alpha^{(i)}$ so that $\varphi^{(0)}$ becomes $\varphi^{(1)}$ (replacing r by $r+1$ if necessary). Let

$$g^{(i)}(x;p) \equiv p_0 - (D_0 \varphi^{(i)}(x) \,/\, D_1 \varphi^{(i)}(x))\, p_1 \; .$$

(Note that $\varphi^{(i)}(0,y) \equiv y^1$ implies that $g^{(i)}$ is analytic on a neighborhood of $\{0\} \times \mathbb{C}^{n+1}$.) Then $\varphi^{(i)}(x)$ is a characteristic function of $g^{(i)}(x;p)$, and $\tau = \alpha^{(i)}(y)$ is the root of $g^{(i)}(0,y;\tau,1,0,\dots,0)$. (For $i = 1$, we define $\alpha^{(1)}(y) \equiv D_0\varphi^{(1)}(0,y)$ if necessary.)

By the definition of an operator of constant multiplicity, we can find k_i such that $h(x;p)[g^{(1)}(x;p)]^{k_1} \dots [g^{(r)}(x;p)]^{k_r} \equiv h_1(x;p)\dots h_s(x;p)$, where for each i, $h_i(x;p)$ is homogeneous in p and $\alpha^{(1)}(y),\dots,\alpha^{(r)}(y)$ are the distinct roots of $h_i(0,y;\tau,1,0,\dots,0)$. Moreover, $\varphi^{(1)},\dots,\varphi^{(r)}$ are characteristic functions of $h_i(x;p)$. Thus, the $h_i(x;p)$ satisfy the conditions of (3).

Now consider the Cauchy problem

$$(9\text{-}2) \qquad a(x;D)[g^{(1)}(x;D)]^{k_1} \dots [g^{(r)}(x;D)]^{k_r} U(x) \equiv v(x)$$

$$(D_0)^j U(0,y) \equiv 0, \; j = 0,\dots, m-1+\Sigma k_i .$$

If $U(x)$ is a solution of this Cauchy problem, then $u(x) \equiv [g^{(1)}(x;D)]^{k_1}\dots[g^{(r)}(x;D)]^{k_r} U(x)$ is a solution of (4-1), with $w^j(y) \equiv 0$. It clearly suffices to prove Parts 1 and 2 (a) of the theorem for (9-2). For Part 2 (b), observe that if $a(x;D)$ has multiplicity one, then $k_1 = 0$ or 1 and $k_i = 0$ for $i > 1$. We thus either leave $a(x;D)$ unchanged, or else change it to an operator of order $m+1$ and then apply an operator of degree one to the solution function $U(x)$ to obtain $u(x)$.

We will show in Section 10 that with the assumptions of Part 2(b), this yields an algebraic function $u(x)$. Since the principal part of the operator in (9-2) is $h(x;p)[g^{(1)}(x;p)]^{k_1} \ldots [g^{(r)}(x;p)]^{k_r}$, this completes our reduction of the problem.

Finally, we note that this reduction also applies to Remark 6.1 Indeed, adding $g^j(y)\log(y^1)$ to $w^j(y)$, $g^j \in \mathcal{A}(\mathcal{O} \cap C^n)$, requires adding the function $g \circ \psi(x) \log[\varphi^{(0)}(x)]$ to the $W^j(x)$ of (9-1). This adds a function of the form $G(x) + h(x)\log[\varphi^{(0)}(x)]$ to the right hand side of (9-1), where $h(x) \in \mathcal{A}(\mathcal{O})$ and $G(x) \in \mathcal{A}(\mathcal{O} - K^{(0)}) \subset \mathcal{A}_q(\mathcal{O} - K^{(0)})$. Thus, it suffices to prove Remark 6.1 for the case $w^j(y) \equiv g^j(y) \equiv 0$. The remark is not affected by the other parts of our reduction.

10. Construction of the Formal Solution

By a <u>formal</u> <u>solution</u> to the Cauchy problem

$$\text{(10-1)} \qquad a(x;D)\,u(x) \equiv v(x)$$

$$(D_0)^j\, u(0,y) \equiv 0, \quad j = 0, \ldots, m-1,$$

we mean a formal sum $u(x) \equiv \Sigma\, u_k(x)$ of functions u_k such that (10-1) is satisfied with term by term differentiation and recombination of terms. If each $u_k \in \mathcal{A}_q(U)$ and the sum is absolutely convergent on U, for some open set U, then $u \in \mathcal{A}_q(U)$ and is an actual solution to the Cauchy problem. Note that we can speak of a formal solution of (10-1) when $v(x)$ is a formal sum. In this case, the first equation of (10-1) denotes term by term equality.

In this section, we construct a formal solution to the Cauchy problem of Theorem I (as reduced in Section 9).

We let f_k denote a sequence of multi-valued analytic functions on $\mathbb{C} - \{0\}$ such that

$$(10\text{-}2) \qquad \frac{df_k}{dt}(t) \equiv f_{k-1}(t)$$

for all integers k. The particular choice of the f_k will be made later.

We utilize the notation of Section 9, so the principle part of $a(x;D)$ is $h_1(x;p) \ldots h_s(x;p)$, where each h_i is of degree r with characteristic functions $\varphi^{(1)}, \ldots, \varphi^{(r)}$. Then $a(x;D)$ has multiplicity one in the direction of $\mathbb{C}^n$ if and only if $s = 1$.

We first prove a result about formal sums.

<u>Lemma 10.1</u>: Let $a(x;D)$ be as in Theorem I, with characteristic functions $\varphi^{(i)}$. Let $u_k^{(i)}(x) \in \mathcal{A}(\mathcal{O})$, with $u_k^{(i)} \equiv 0$ for $k < 0$. Then

(1) There exist operators $\mathcal{L}_j^{(i)}(x;D)$ on $\mathcal{O}$ of degree

$\leqq j$, for $j = 1, \ldots, m$, such that

$$(10\text{-}3) \qquad a(x;D)\Big(\sum_{i=1}^{r}\sum_{k=0}^{\infty} u_k^{(i)}(x)\, f_{k-\ell}[\varphi^{(i)}(x)]\Big)$$

$$\equiv \sum_{i=1}^{r}\sum_{k=0}^{\infty}\Big(\sum_{j=1}^{m} \mathcal{L}_j^{(i)}[u_{k+1-j}^{(i)}(x)]\Big) f_{k-\ell-m+1}[\varphi^{(i)}(x)]$$

(2) $\mathcal{L}_j^{(i)}(x;D)$ depends only on $a(x;D)$ and $\varphi^{(i)}(x)$.

(3) The principal part of $\mathcal{L}_1^{(i)}(x;D)$ is

$$\sum_{j=0}^{n} \frac{\partial h}{\partial p_j}(x;D\varphi^{(i)}(x))\, p_j$$

where $h(x;p)$ is the principal part of $a(x;D)$.

<u>Proof</u>: Note that the left hand side of (10-3) denotes term by term differentiation. Performing the differentiation, it is clear from (10-2) that we can find operators $\mathcal{L}_j^{(i)}(x;D)$ on $\mathcal{O}$ of degree $\leqq j$, for $j = 0, \ldots, r$, such that

$$(10\text{-}4) \qquad a(x;D)(\ldots) = \sum_i \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{m} \mathcal{L}_j^{(i)}[u_{k-j}(x)]\Big) f_{k-\ell-m}[\varphi^{(i)}(x)] \,.$$

Moreover, (2) is satisfied by the $\mathcal{L}_j^{(i)}$.

An elementary calculation shows that

$$(10\text{-}5) \qquad h(x;D)(u_k(x)\, f_{k-\ell}[\varphi^{(i)}(x)])$$

$$\equiv h(x;D\varphi^{(i)}(x))\, u_k(x)\, f_{k-m}[\varphi^{(i)}(x)]$$

$$+ \sum_{j=0}^{n} \frac{\partial h}{\partial p_j} (x; D\varphi^{(i)}(x))\, D_j\, u_k(x)\, f_{k-m+1}[\varphi^{(i)}(x)]$$

$$+ \sum_{s=2}^{r} b_s(x;D)[u_k(x)]\; f_{k-m+s}[\varphi^{(i)}(x)]$$

for some operators $b_s(x;D)$ of order $\leqq p$. But $h(x;D\varphi^{(i)}(x)) \equiv 0$, since $\varphi^{(i)}$ is a characteristic function of $h(x;p)$. From (10-5), it is then easy to see that $\mathcal{L}_0^{(i)}(x;D) \equiv 0$. Substituting $k+1$ for k in (10-4) then gives (10-3).

It also follows from (10-5) that

$$\mathcal{L}_1^{(0)}(x;D) \equiv \sum_{j=0}^{n} \frac{\partial h}{\partial p_j} (x;\varphi^{(i)}(x))\, D_j + w(x)$$

for some function $w(x)$, proving (3). $\blacksquare$

The fact that $\mathcal{L}_0^{(i)} \equiv 0$ allows us to construct a formal solution for (10-1) as follows.

<u>Lemma 10.2</u>: Let $a(x;D)$ have multiplicity one in the direction of $\mathbb{C}^{n-1}$, and let the $\mathcal{L}_j^{(i)}$ be as in Lemma 10.1. There exists a neighborhood U of 0 in $\mathbb{C}^{n+1}$ and operators $\mathfrak{M}_j^{(i)}(x;D)$ on U of degree $\leqq j$ with the following property:

Let $v(x) \equiv \sum_{i=1}^{r} \sum_{k=0}^{\infty} v_k^{(i)}(x)\, f_{k-\ell}[\varphi^{(i)}(x)]$, with each $v_k^{(i)} \in \mathcal{A}(U)$, and let $v_k^{(i)} \equiv 0$ for $k < 0$. Then

(1) There exist functions $u_k^{(i)}(x) \epsilon \mathcal{A}(U)$ such that

$$u(x) \equiv \sum_{i=1}^{r} \sum_{k=0}^{\infty} u_k^{(i)}(x)\, f_{k-\ell+r-1}[\varphi^{(i)}(x)]$$

is a formal solution of (10-1).

(2) The $u_k^{(i)}(x)$ are solutions to the following sequence of Cauchy problems:

$u_k^{(i)} \equiv 0$ for $k < 0$

(10-6) (a) $$\mathcal{L}_1^{(i)}[u_0^{(i)}(x)] \equiv v_0^{(i)}(x)$$

(b) $$u_0^{(i)}(0,y) \equiv 0$$

(10-7) (a) $$\mathcal{L}_1^{(i)}[u_k^{(i)}(x)] \equiv v_k^{(i)}(x) - \sum_{q=2}^{r} \mathcal{L}_q^{(i)}[u_{k+1-q}^{(i)}(x)]$$

(b) $$u_k^{(i)}(0,y) \equiv \sum_{j=1}^{r} \sum_{q=1}^{r-1} \mathcal{M}_q^{(j)}[u_{k-q}^{(j)}](0,y) \, ,$$

for $k > 0$.

Proof: It follows from Lemma 10.1 that if the $u_k^{(i)}$ satisfy (10-6) (a) and (10-7) (a), then

$$a(x;D)\, u(x) \equiv v(x) \; .$$

(Recall that $m = r$, since $a(x;D)$ has multiplicity one.) We now construct the operators $\mathcal{M}_q^{(i)}(x;D)$ so that (10-6) (b) and (10-7) (b) imply

$$(D_0)^j u(0,y) \equiv 0 \quad , \quad j = 0, \dots, m-1 \; .$$

Term by term differentiation shows that we can find operators $\mathcal{M}_{q,j}^{(i)}(x;D)$ of order $\leqq q$ such that

$$(10\text{-}8) \qquad (D_0)^j u(x) \equiv \sum_{i=1}^{r} \sum_{k=0}^{\infty} \Big(\sum_{q=0}^{j} \mathcal{M}_{q,j}^{(i)} [u_{k-q}(x)]\Big) f_{k-\ell+r-1-j}[\varphi^{(i)}(x)] \; .$$

Moreover,

$$(10\text{-}9) \qquad \mathcal{M}_{0,j}^{(i)}(x;D) \equiv [D_0 \varphi^{(i)}(x)]^j \; .$$

Recall that $\varphi^{(i)}(0,y) \equiv y^1$ and $D_0 \varphi^{(i)}(0,y) \equiv \alpha^{(i)}(y)$. Then from (10-8) and (10-9) we see that $(D_0)^j u(0,y) \equiv 0$ if

$$(10\text{-}10) \qquad \sum_{i=1}^{r} [\alpha^{(i)}(y)]^j u_k^{(i)}(0,y) \equiv \sum_{i=1}^{r} \sum_{q=1}^{j} \mathcal{M}_{q,j}^{(i)} [u_{k-q}^{(i)}](0,y)$$

for each k .

Let $\Delta(y)$ be the Vandermonde determinant

$$\Delta(y) \equiv \det([\alpha^{(i)}(y)]^j) \; .$$

Since the $\alpha^{(i)}(0)$ are distinct, $\Delta(y) \neq 0$ in a neighborhood of 0 in $\mathbb{C}^n$. We can thus solve (10-10) for the $u_k^{(i)}(0,y)$ by Cramer's rule, getting equation (10-7) (b) for some operators $\mathfrak{M}_q^{(j)}(y;D)$ on a neighborhood of 0 in $\mathbb{C}^n$. The $\mathfrak{M}_q^{(j)}$ depend only on $a(x;D)$ and the $\varphi^{(i)}(x)$, and are independent of k. We can then extend the $\mathfrak{M}_q^{(j)}$ to operators on a neighborhood N of 0 in $\mathbb{C}^{n+1}$. Hence, (10-7) (b) implies that $(D_0)^j u(0,y) \equiv 0$ for $j = 0, \ldots, m-1$, so $u(x)$ is a formal solution of (10-1). (Note that (10-6) is a special case of (10-7), since $u_k \equiv 0$ for $k < 0$.)

Finally, by the Cauchy-Kowalewski Theorem, we can find a neighborhood U of 0 contained in N such that for each i, if $V(x) \in \mathcal{A}(U)$ and $W(y) \in \mathcal{A}(U \cap \mathbb{C}^n)$, then the Cauchy problem

$$\mathcal{L}_1^{(i)}[u](x) \equiv V(x)$$

$$u(0,y) \equiv W(y)$$

has a solution $u(x) \in \mathcal{A}(U)$. (For the operators $\mathcal{L}_1^{(i)}$ of order one, this follows from Lemmas 12.4 and 12.5.) Thus, (10-6) and (10-7) inductively define the $u_k^{(i)}(x) \in \mathcal{A}(U)$. ∎

For the general case of an operator $a(x;D)$ of constant multiplicity, we proceed as follows. By assumption (3) of Section 9, we can write

$$a(x;D) \equiv h_1(x;D) \ldots h_s(x;D) + b(x;D) ,$$

with degree of $b(x;D) < m = rs$. We then apply Lemma 10.2 s times to inductively find the formal solutions ${}_k u(x)$ satisfying

$$(10\text{-}11) \qquad h_1(x;D) \ldots h_s(x;D)\,[{}_k u(x)] \equiv \begin{cases} v(x) & \text{if } k = 0 \\ -b(x;D)\,[{}_{k-1}u(x)] & \text{if } k > 0 \end{cases}$$

$$(D_0)^j\,[{}_k u]\,(0,y) \equiv 0 .$$

Then $\sum_{k=0}^{\infty} {}_k u(x)$ is easily seen to be a formal solution to (10-1).

We now show that the reduction of Section 9 applies to Part 2 (b) of the Theorem. We will choose the f_k so that if $k \geqq 0$, then f_k is an algebraic function on $\mathbb{C}$ with singularity at 0. Then the sum

$$(10\text{-}12) \qquad \sum_{i=1}^{r} \sum_{k=0}^{\infty} u_k^{(i)}(x)\, f_k\,[\varphi^{(i)}(x)]$$

is algebraic on a neighborhood of 0 in $\mathbb{C}^{n+1}$, if it converges. For Part 2 (b) of the theorem, we will write the functions $v(x)$ and $W^j(x)$ of (9-1) in the form

$$\sum_{k=0}^{\infty} v_k(x)\, f_k\,[\varphi^{(1)}(x)] .$$

Then by Lemma 10.1, the right hand side of (9-1) can be written in the form

$$\sum_{k=0}^{\infty} v_k(x)\, f_{k-r+1}\,[\varphi^{(1)}(x)] .$$

Lemma 10.2 then gives a solution function of the form (10-12), which is algebraic. This shows that we can assume $w^j(y) \equiv 0$ in the proof of Part 2 (b) of Theorem I.

If $a(x;D)$ is replaced by $a(x;D)\, g^{(1)}(x;D)$, then Lemma 10.2 gives a solution

$$U(x) \equiv \sum_{i=1}^{r} \sum_{k=0}^{\infty} u_k^{(i)}(x)\, f_{k+1}[\varphi^{(i)}(x)]$$

of (9-1). Then $u(x) \equiv g^{(1)}(x;D)\, U(x)$ has the form of (10-12), and is algebraic. This finishes the proof that it suffices to prove Theorem I with assumption (3) of Section 9.

11. Notations

We assume that we have a star-like domain $\mathcal{D}$ with $\mathcal{O} \subset \mathcal{D} \times \mathbb{C}^n$. Let η be as in Definition 8.2. For any $t \in \mathcal{D}$, we define $\|t\|$ by

$$\|t\| = |\eta^{-1}(t)| .$$

We define the real path η_θ in $\mathcal{D}$, $0 \leqq \theta < 2\pi$, by

$$\eta_\theta(\rho) \equiv (\rho e^{i\theta}) .$$

Note that for any $t \in \mathcal{D}$, $t = \eta_\theta(\|t\|)$ for some θ.

For the proof of Theorem I, we will let $\mathcal{D} = \mathbb{C}$ and η be the identity mapping. Then $\|t\| = |t|$, and $\eta_\theta(\rho) \equiv \rho e^{i\theta}$. We consider the more general case so that we may apply some results from this chapter to the proof of Theorem IV.

We now define the functions $E(r,s)$ on $\mathcal{D} \times \mathbb{C}^n$, for non-negative integers r and s, by

(11-1) $$E(r,s)(t,y) \equiv \frac{(r+s)!}{\rho^{r+s}} \exp(\gamma \|t\|) [\exp(\gamma n \|t\|)$$

$$(1 + \gamma n \|t\|)]^{r+s} (\gamma n)^r ,$$

for some constants $\gamma, \rho > 0$. Observe that $E(r,s)(x)$ is an increasing function of r, s, γ and $1/\rho$. Also notice that

(11-2) $$E(r, s+j) \leqq E(r+j, s),$$

if $\gamma \geqq 1/n$.

For any number $T \geqq 0$, we can choose E_T such that

$$E_T \geqq [\exp(\gamma n \|t\|) (1 + \gamma n \|t\|)] \frac{(\gamma n)}{\rho} \exp(\gamma \|t\|)$$

for all $t \in \mathcal{D}$ with $|t| \leqq T$. We then have

(11-3) $$E(r+1,s)(t,y) \leqq (r+1+s) E_T E(r,s)(t,y)$$

(11-4) $$E(r,0) \leqq r! (E_T)^r$$

whenever $|t| \leqq T$. (Like $E(r,s)$, E_T depends upon ρ and γ.)

Let $\mathbb{N}$ denote the set of non-negative integers, and $\mathbb{N}^k$ the set of k-tuples of elements of $\mathbb{N}$. For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we define

$$|\alpha| = \alpha_1 + \ldots + \alpha_k .$$

If $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, then we define

$$D^\alpha f \equiv (D_0)^{\alpha_0} \ldots (D_n)^{\alpha_n} f$$

Note that D^α is an operator of order $|\alpha|$. Similarly, for $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we let

$$D^\beta f \equiv (D_1)^{\beta_1} \ldots (D_n)^{\beta_n} f .$$

We now state some simple relations which we will need later on. The proofs are elementary, and are omitted. Assume $j, k \in \mathbb{N}$. Then

$$(11\text{-}5) \qquad (j + k)! \geqq j!\, k! .$$

If $j \leqq k$, then

$$(11\text{-}6) \qquad \frac{k!}{j!\,(k-j)!} \leqq 2^k$$

$$(11\text{-}7) \qquad \sum_{i=k}^{\infty} \frac{i!}{(i-j)!} \rho^i \leqq j! \frac{\rho^j}{(1-\rho)^{j+1}}$$

for any $\rho \in R$ with $0 \leqq \rho < 1$. (To verify (11-7), write $1/(1-\rho) = \Sigma \rho^i$ and differentiate j times.)

Finally, for $k < 0$ we define $k!$ to equal $1/(-k)!$. We then have

$$(11-8) \qquad k! \leqq (k+1)!$$

for all integers k .

12. Mizohata's Lemmas

In order to establish the convergence of our formal solution, we will need bounds on the solutions $u_k(x)$ of (10-6) and (10-7). For this, we will use some results of Mizohata [8].

First, we prove the following simple result.

Lemma 12.1: Let K be a compact subset of $\mathcal{O}$. There exists a constant ρ such that if $w(x)$ is any bounded analytic function on $\mathcal{O}$, then

$$|D^\alpha w(x)| \leqq M(|\alpha|!)/\rho^{|\alpha|}$$

for all $x \in K$ and $\alpha \in \mathbb{N}^{n+1}$; where $M = \sup_{x \in \mathcal{O}} |w(x)|$.

Proof: Let $N(x;\rho) = \{z \in \mathbb{C}^{n+1} : |x^i - z^i| < \rho;\ i = 0, \ldots, n\}$. One easily deduces from the $(n+1)$ - dimensional Cauchy integral formula that if $N(x;\rho) \subset \mathcal{O}$, then

$$|D^{\alpha} w(z)| \leqq M \frac{\alpha_0! \ldots \alpha_n!}{\rho^{|\alpha|}}$$

for all $z \in N(x;\rho)$, and all $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$.

By the compactness of K, we can find points $x_i \in K$ and ρ_i such that

$$K \subset N(x_1;\rho_1) \cup \ldots \cup N(x_m;\rho_m) \subset \mathcal{O}.$$

Choosing ρ to be the smallest of the ρ_i proves the lemma, since $\alpha_0! \ldots \alpha_n! \leqq |\alpha|!$. ∎

We now state the first of Mizohata's lemmas, which is Lemma 1 of [8].

<u>Lemma 12.2</u>: Let $f, g \in \mathcal{A}(\mathcal{O})$, $x \in \mathcal{O}$, and assume that

$$|D^{\alpha} f(x)| \leqq \frac{(r + |\alpha|)!}{(k\rho)^{|\alpha|}} F, \quad k > 1$$

$$|D^{\alpha} g(x)| \leqq \frac{(s + |\alpha|)!}{\rho^{|\alpha|}} G$$

for all $\alpha \in \mathbb{N}^{n+1}$, where r and s are non-negative integers. Then

$$|D^{\alpha}(fg)(x)| \leqq \frac{k}{k-1} \frac{(r+s+|\alpha|)!}{\rho^{|\alpha|}} \frac{FG}{\binom{r+s}{r}} .$$

Using these two lemmas, we prove the following.

Lemma 12.3: Let $L(x;D)$ be an operator of degree $\leqq k$ on $\mathcal{O}$. Let U be an open subset of $\mathcal{O}$ with compact closure contained in $\mathcal{O}$. Then for sufficiently large constants γ, $1/\rho$ and K, depending only on $L(x;D)$ and U:

If $w \in \mathcal{A}(U)$ and satisfies

$$|D_0^q\ D^\beta w(x)| \leqq A\,E\,(s+q,\ |\beta|)(x)$$

for some $s \in \mathbb{N}$ and all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$; then $L\lfloor w\rfloor$ satisfies

$$|D_0^q\ D^\beta L\lfloor w\rfloor(x)| \leqq K\,A\,E\,(s+k+q,\ |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

Proof: First suppose $L(x;D) = D_0^i\ D^\upsilon$ for some $\upsilon \in \mathbb{N}^n$, with $i + |\upsilon| \leqq k$. Then the result is true with $K = 1$, since

$$|D_0^q\ D^\beta L[w](x)| = |D_0^{q+i}\ D^{\beta+\upsilon}\ w(x)|$$

$$\leqq A\,E\,(s+i+q,\ |\beta| + |\upsilon|)(x) \quad \text{[by hypothesis on } w]$$

$$\leqq E\,(s+i+|\upsilon|+q,\ |\beta|)(x) \quad \text{[by (11-2)]}$$

$$\leqq E\,(s+k+q,\ |\beta|)(x) \quad [\text{since } i + |\upsilon| \leqq k].$$

Next, suppose L is multiplication by a function $f(x)$, so L has order 0. By Lemma 12.1, for sufficiently large constants

F and $1/\rho$, depending only on f and U,

$$|D^{\alpha} f(x)| \leqq \frac{F(|\alpha|!)}{(2\rho)^{|\alpha|}}$$

for all $x \in U$. Writing $E(s+q, |\beta|)(x) = (s+q+|\beta|)!\, G/\rho^{q+|\beta|}$ and applying Lemma 12.2 gives the desired result, with $K = F$.

Finally, write $L(x;D) \equiv \Sigma f_{i,\upsilon}(x)\, D_0^i\, D^{\upsilon}$. Applying the above results for the operators $f_{i,\upsilon}(x)$ and $D_0^i\, D^{\upsilon}$ easily proves the lemma. $\blacksquare$

We now consider the first order operator $\mathcal{L}$ defined by

$$\mathcal{L}(x;D) \equiv D_0 + \sum_{i=1}^{n} a_i(x)\, D_i + b(x) \,, \tag{12-1}$$

where $a_i(x), b(x) \in \mathcal{A}(\mathcal{O})$. We wish to examine the solution of the Cauchy problem

$$\begin{aligned} &\mathcal{L}[u](x) \equiv f(x) \\ &u(0,y) \equiv g(y) \,. \end{aligned} \tag{12-2}$$

For each $y \in \mathbb{C}^n \cap \mathcal{O}$, let γ_y be the complex path in $\mathbb{C}^n$ defined by

$$\begin{aligned} &\frac{d\gamma_y}{dt}(t) \equiv (a_1(t, \gamma_y(t)), \ldots, a_n(t, \gamma_y(t))) \,. \\ &\gamma_y(0) = y \,. \end{aligned} \tag{12-3}$$

Define the mapping Γ on a subset of $\mathbb{C}^{n+1}$ by

$$\Gamma(t,y) \equiv (t, \gamma_y(t)) \,.$$

From the theory of ordinary differential equations, we know that Γ is a 1-1 analytic mapping with an analytic inverse. Moreover, a simple calculation shows that

$$\mathscr{L}[u]\circ\Gamma \equiv D_0 [u\circ\Gamma] + (b\circ\Gamma)(u\circ\Gamma).$$

Hence, the solution to (12-2) can be obtained from

$$\text{(12-4)} \quad u\circ\Gamma (t,y) \equiv [\exp(-\int_0^t b\circ\Gamma (r,y)\, dr)] [\int_0^t f\circ\Gamma (s,y) \cdot \exp(\int_0^s b\circ\Gamma (r,y)\, dr)\, ds + g(y)],$$

where the integration is along some path in $\mathbb{C}$. We will perform the integration along the paths η_θ.

Now let U be an open subset of $\mathcal{O}$ and assume $f(x) \in \mathcal{A}(U)$, $g(y) \in \mathcal{A}(U \cap \mathbb{C}^n)$. In order to insure that we can perform the integration in (12-4) to get a solution $u(x) \in \mathcal{A}(U)$, we make the following assumptions about U:

(1) If $x \in U$, then $x = \Gamma(t,y)$ for some $t \in \mathcal{S}$, $y \in U \cap \mathbb{C}^n$.

(2) If $\Gamma(\eta_\theta(\tau), y) \in U$, then $\Gamma(\eta_\theta(\rho\tau), y) \in U$ whenever $0 \leqq \rho \leqq 1$.

In order to obtain our bounds on $|u(x)|$, we make the following additional assumptions:

(3) U has compact closure contained in $\mathcal{O}$. We can then choose T such that $(t,y) \in U$ implies $|t| \leqq T$.

(4) If $(\eta_\theta(\rho), y) \in U$, then $\left|\frac{d\eta_\theta}{d\rho}(\rho)\right| \leqq 1$.

Note that (4) is always satisfied if η is the identity mapping.

We can now state our result, which combines and generalizes Propositions 3 and 4 of [8]. The proof is given in the appendix.

<u>Lemma 12.4</u>: Let $\mathcal{L}$, U, η and T be as above. For sufficiently large constants γ and $1/\rho$, depending only on $\mathcal{L}$ and U:

<u>If:</u> (a) $f \in \mathcal{A}(U)$ and for some A, $r \geqq 0$ satisfies

$$|D_0^q \; D^\beta f(x)| \leqq A\, E\,(r+1+q, |\beta|)\,(x)$$

for all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$

(b) $g \in \mathcal{A}(U \cap \mathbb{C}^n)$ and for some $B \geqq 0$ satisfies

$$|D^\beta g(y)| \leqq B\, E\,(r, |\beta|)\,(0,y)$$

for all $y \in U \cap \mathbb{C}^n$ and $\beta \in \mathbb{N}^n$.

<u>Then:</u> There exists a solution $u(x) \in \mathcal{A}(U)$ of the Cauchy problem (12-2) satisfying

$$|D_0^q \; D^\beta \; u(x)| \leqq 2\,(A\,E_T + B)\, E\,(r+q, |\beta|\;(x)$$

for all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

To be able to apply Lemma 12.4 to the operators $\mathcal{L}_1^{(i)}$ of (10-6), we need the following result.

<u>Lemma 12.5</u>: Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be operators of the form (12-1), and let η be the identity mapping. There exists a single open subset of U of $\mathcal{O}$, with $0 \in U$, satisfying conditions (1) - (3) above for each $\mathcal{L}_i$.

<u>Proof</u>: Choose a compact subset K of $\mathcal{O}$. Let be given by (12-1) and choose M such that

(12-5) $\quad |a_i(x)| \leqq M$

for all $x \in K$, and $i = 1, \ldots, n$. Let $s, t \in \mathbb{C}$ and suppose that γ_y maps the straight line segment between s and t into K. It follows from (12-3) that if $\gamma_y(t) = (z^1, \ldots, z^n)$ and $\gamma_y(s) = (w^1, \ldots, w^n)$, then

(12-6) $\quad |z^i - w^i| \leqq M\,|t - s|$

for each $i = 1, \ldots, n$.

Now let Γ_i be the mapping Γ defined above for the operator $\mathcal{L}_i$. Let V be an open neighborhood of 0 in $\mathbb{C}^n$ with compact closure contained in $\mathcal{O}$. We can choose an $\epsilon > 0$ such that for each i, $\Gamma_i(t,y) \in \mathcal{O}$ whenever $y \in V$ and $|t| < \epsilon$. We can also choose an [(n+1) - dimensional] open neighborhood N of V in $\mathcal{O}$ such that for each $x \in N$ and each i, $x = \Gamma_i(t,y)$ for some $y \in \mathcal{O} \cap \mathbb{C}^n$ and $t \in C$ with $|t| < \epsilon$. Moreover, we can assume that $N \subset K$ for some compact subset K of $\mathcal{O}$.

Choose M so that (12-5) is satisfied for each of the $\mathcal{L}_i$. For any $y \in C^n$, $\delta > 0$, define

$$K(y;\delta) = \{ (t,z) : |t| < \delta/M \text{ and } |z^i - y^i| < \delta - M|t| ; \text{ for } i = 1, \ldots, n \}.$$

Then $K(y;\delta)$ is an open subset of $\mathbb{C}^{n+1}$, and $K(y;\delta) \cap \mathbb{C}^n = \{z : |z^i - y^i| < \delta\}$.

Assume that $K(y_0, \delta) \subset N$. We will show that for each i, if $\Gamma_i(t,y) \in K(y_0;\delta)$ and $0 \leqq \rho \leqq 1$, then $\Gamma_i(\rho t, y) \in K(y_0;\delta)$. We can write $\Gamma_i(t,y) = (t, \gamma_y(t))$. Let $\gamma_y(t) = z$ and $\gamma_y(\rho t) = w$. By (12-6), we have

$$|z^i - w^i| \leqq M(1-\rho)|t|.$$

Since $(t,z) \in K(y_0;\delta)$, we have

$$|y_0^i - z^i| < \delta - M|t|.$$

Adding these inequalities yields

$$|y_0^i - w^i| < \delta - M|\rho t|,$$

which shows that $\Gamma_i(t,y) = (t,w) \in K(y_0;\delta)$.

For each $y \in V$, choose $\delta_y > 0$ such that $K(y;\delta_y) \subset N$ and $K(y;\delta_y) \cap \mathbb{C}^n \subset V$. Since N and V are open sets, we can obviously do this. Let $U = \bigcup_{y \in V} K(y;\delta_y)$.

Since $U \subset N$, condition (1) is satisfied for each i. Since $\eta_\theta(\rho\tau) = \rho\,\eta_\theta(\tau)$, condition (2) follows from the fact that $\Gamma(t,y) \in K(y_0;\delta)$ implies $\Gamma(\rho t, y) \in K(y_0;\delta)$ if $0 \leqq \rho \leqq 1$. Finally, condition (3) is obviously satisfied because $U \subset N \subset K$. ∎

13. Bounds on the Formal Solutions

We now establish bounds on the formal solutions obtained in Section 10, which will be used to prove their convergence. We first obtain bounds for the functions $u_k^{(i)}$ of Lemma 10.2.

Lemma 13.1: Let $a(x;D)$, U, $v(x)$ and $u(x)$ be as in Lemma 10.2. We can choose the neighborhood U and a constant B such that for all sufficiently large C, γ and $1/\rho$.

If: for some constants $A, r \geqq 0$, each $v_k^{(i)}$ satisfies

$$|D_0^q\, D^\beta\, v_k^{(i)}(x)| \leqq A\, C^k\, E(r+1+k+q, |\beta|)(x)$$

for each $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$,

Then: each $u_k^{(i)}$ satisfies

$$|D_0^q\, D^\beta\, u_k^{(i)}(x)| \leqq A\, B\, C^k\, E(r+k+q, |\beta|)(x) \tag{13-1}$$

for each $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

The choice of U, B, C, γ and ρ depends only on $a(x;D)$, and B may be chosen to be arbitrarily large. Moreover, a single such choice can be made for any finite collection of operators $a(x;D)$.

<u>Proof:</u> Since $a(x;\rho)$ has multiplicity one, $\frac{\partial h}{\partial p_o}(0; D\varphi^{(i)}(0)) = \frac{\partial h}{\partial p_o}(0, \alpha^{(i)}(0)) \neq 0$. Replace $\mathcal{O}$ by a smaller neighborhood of 0, if necessary, so that $\frac{\partial h}{\partial p_o}(x; D\varphi^{(i)}(x)) \neq 0$ for all $x \in \mathcal{O}$, $i = 1, \ldots, r$. Let $h^{(i)}(x) \equiv (1/\frac{\partial h}{\partial p_o}(x; D\varphi^{(i)}(x)))$.

Now let $\mathcal{L}_1^{(i)}(x; D)$ be the operator defined in Lemma 10.1, and let $\bar{\mathcal{L}}_1^{(i)}(x; D) \equiv h^{(i)}(x)\mathcal{L}_1^{(i)}(x; D)$. Then part (3) of Lemma 10.1 implies that $\bar{\mathcal{L}}_1^{(i)}$ has the form of (12-1). The Cauchy problems (10-6) and (10-7) can be replaced by the following equivalent ones:

$$(13\text{-}2) \quad \text{(a)} \quad \bar{\mathcal{L}}_1^{(i)}[u_o^{(i)}(x)] \equiv h^{(i)}(x)v_o^{(i)}(x)$$

$$\text{(b)} \quad u_o^{(i)}(0, y) \equiv 0$$

$$(13\text{-}3) \quad \text{(a)} \quad \bar{\mathcal{L}}_1^{(i)}[u_k^{(i)}(x)] \equiv h^{(i)}(x)[v_k^{(i)}(x) - \sum_{s=2}^{r} \mathcal{L}_s^{(i)}[u_{k+1-s}^{(i)}(x)]$$

$$\text{(b)} \quad u_k^{(i)}(0, y) \equiv \sum_{j=1}^{r} \sum_{s=1}^{r-1} \mathcal{M}_s^{(j)}[u_{k-s}^{(j)}](0, y).$$

Replace $\mathcal{O}$ by the set U of Lemma 10.2, which we can assume to have compact closure in $\mathcal{O}$, and apply Lemma 12.5 to the operators $\bar{\mathcal{L}}_1^{(i)}$ of Lemma 10.2 to choose U. Then Lemma 12.4 can be applied to the Cauchy problems (13-2) and (13-3). For a finite collection of operators $a(x; D)$, we replace $\mathcal{O}$ by the intersection of the U's of

Lemma 10.2, and apply Lemma 12.5 to all the $\bar{\mathcal{L}}_1^{(i)}$. Observe that Lemma 10.2 remains valid if U is replaced by any smaller set which satisfies the hypotheses of Lemma 12.4.

We now prove (13-1) by induction on k. Choose K to be large enough to satisfy the conclusion of Lemma 12.3 for each of the $o\underline{th}$ degree operators $h^{(i)}(x)$. Applying Lemma 12.3 to the operator $h^{(i)}(x)$ and the function $v_o^{(i)}(x)$, using the hypothesis on $v_o^{(i)}$, and then applying Lemma 12.4 (with B = 0) to (13-2) yields (13-1) for k = 0, if $B \geq 2E_T K$.

Now assume $k \geq 1$ and (13-1) holds for all $u_j^{(i)}$ with $j < k$. Considering the right hand side of (13-3) (a), we have

$$|D_o^q D^\beta (v_k^{(i)}(x) - \sum_{s=2}^{r} \mathcal{L}_s^{(i)}[u_{k+1-s}^{(i)}(x)])|$$

$$\leq |D_o^q D^\beta v_k^{(i)}(x)| + \sum_{s=2}^{r} |D_o^q D^\beta (\mathcal{L}_s^{(i)}[u_{k+1-s}^{(i)}(x)])| .$$

But

$$|D_o^q D^\beta v_k^{(i)}(x)| \leq A C^k E(r+1+k+q, |\beta|)(x)$$

by hypothesis. By the induction assumption for $u_{k+1-s}^{(i)}$, and Lemma

12.3 applied to $\mathcal{L}_s^{(i)}$, we get

$$|D_o^q D^\beta \mathcal{L}_s^{(i)}[u_{k+1-s}^{(i)}(x)]| \leq ABC^{k+1-s} L_s^{(i)} E(r+1+k+q, |\beta|)(x)$$

for some constant $L_s^{(i)}$ depending only on $\mathcal{L}_s^{(i)}$ - hence depending only on $a(x; D)$. If we choose $C \geq B \sum_{s=2}^{r} L_s^{(i)}$ and $C \geq 1$, we get

$$\sum_{s=2}^{r} B C^{k+1-s} L_s^{(i)} \leq C^k.$$

Combining this with the above inequalities, we get

$$(13\text{-}4) \qquad |D_o^q D^\beta (v_k^{(i)}(x) - \sum_{s=2}^{r} \mathcal{L}_s^{(i)}[u_{k+1-s}^{(i)}(x)])|$$

$$\leq 2AC^k E(r+1+k+q, |\beta|)(x).$$

Applying Lemma 12.3, using (13-4), we get

$$(13\text{-}5) \qquad |D_o^q D^\beta [h^{(i)}(x)(v_k^{(i)}(x) - \sum_{s=2}^{r} \mathcal{L}_s^{(i)}[u_{k+1-s}^{(i)}(x)])]|$$

$$\leq 2AKC^k E(r+1+k+q, |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$, $\beta \in \mathbb{N}^n$.

For the right hand side of (13-3) (b), we apply the induction hypothesis to $u_{k-s}^{(j)}$ and Lemma 12.3 to $\mathfrak{m}_s^{(j)}$, getting

$$|D^\beta(\sum_{j=1}^{r}\sum_{s=1}^{r-1} \mathfrak{m}_s^{(j)}[u_{k-s}^{(j)}](0,y))|$$

$$\leq \sum_{j=1}^{r}\sum_{s=1}^{r-1} A B C^{k-s} M_s^{(j)} E(r+k, |\beta|)(0,y)$$

for some constants $M_s^{(j)}$ depending only on $a(x;D)$. If $C \geq B \sum_{s,j} M_s^{(j)}$ and $C \geq 1$, this gives

$$(13\text{-}6) \qquad |D^\beta(\sum_{j=1}^{r}\sum_{s=1}^{r-1} \mathfrak{m}_s^{(j)}[u_{k-s}^{(j)}](0,y))|$$

$$\leq A C^k E(r+k, |\beta|)(0,y).$$

We can now use (13-5) and (13-6) to apply Lemma 12.4 to the Cauchy problem (13-3), getting

$$|D_o^q D^\beta u_k^{(i)}(x)| \leq 2A(2E_T K + 1)C^k E(r+k+q, |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^{n+1}$. Requiring that $B \geq 2(2E_T K+1)$ and then choosing C as required by the above inequalities, completes the proof of (13-1). Note that B can be chosen arbitrarily large.

Observe that in the above proof, the choice of the constants γ and ρ for the functions $E(i, j)$ depended only on $a(x; D)$. Moreover, γ and $1/\rho$ can always be increased.

It is clear from the proof that we can make a single choice of B, C, ρ and γ for any finite collection of operators $a(x; D)$. ∎

In order to find bounds for the solutions ${}_k u(x)$ of (10-11), we will need the following two results.

<u>Lemma 13.2</u>: Let $h_1(x; p), \ldots, h_s(x; p)$ be as in Section 9. Then there exist a neighborhood U of o and a constant B such that for any sufficiently large C, γ and $1/\rho$:

<u>If</u>: $v(x) \equiv \sum_{i=1}^{r} \sum_{k=0}^{\infty} v_k^{(i)}(x) f_{k-\ell}[\varphi^{(i)}(x)]$,

where each $v_k^{(i)} \in \mathcal{O}(U)$ and satisfies

$$|D_o^q D^\beta v_k^{(i)}(x)| \le A\, C^k E(r+s+k+q, |\beta|)(x)$$

for some constants A, $r \ge 0$ (independent of k and i) and all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$,

<u>Then</u>: The Cauchy problem

(13-7) $h_1(x; D) \ldots h_s(x; D)u(x) \equiv v(x)$

$$(D_o)^j u(0, y) \equiv 0 \quad , \quad j = 0, \ldots, m-1; \quad m = rs,$$

has a formal solution

$$u(x) \equiv \sum_{i=1}^{r} \sum_{k=0}^{\infty} u_k^{(i)}(x) f_{k-\ell+m-s}[\varphi^{(i)}(x)] ,$$

where each $u_k^{(i)} \in \mathcal{A}(U)$ and satisfies

(13-8) $$|D_o^q D^\beta u_k^{(i)}(x)| \leq A B C^k E(r+k+q, |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

The choice of U, B, C, γ and ρ depends only on the h_i, and B can be chosen arbitrarily large.

Proof: Consider the Cauchy problems

(13-9) $h_j(x; D)u_j(x) \equiv u_{j-1}(x)$

$$(D_o)^i u_j(0, y) \equiv 0 \quad , \quad i = 0, \ldots, r-1,$$

for $j = 1, \ldots, s$; where $u_o \equiv v$. Then $u(x) \equiv u_k(x)$ formally satisfies

$$h_1(x;D)\ldots h_s(x;D)u(x) \equiv v(x).$$

Applying $(D_o)^{k-r}$ to the first of equations (13-9), and using the fact that the coefficient of $(D_o)^r$ in $h_j(0;D)$ is not zero (since $h(0;\delta^o) \neq 0$), gives a simple inductive proof that $(D_o)^k u_j(0,y) \equiv 0$ for all $k = 0,\ldots,jr-1$. This shows that $u(x) \equiv u_k(x)$ is a solution to the Cauchy problem (13-7).

Applying Lemma 13.1 successively to the s Cauchy problems (13-9) gives the required result (since $s(r-1) = m-s$), where U, γ and ρ are as in Lemma 13.1, and B and C are the $s^{\underline{th}}$ roots of the B and C of that lemma. ∎

<u>Lemma 13.3</u>: Let $L(x,D)$ be an operator on $\mathcal{O}$ of degree $\leq j$, $\varphi \in \mathcal{A}(\mathcal{O})$, and U an open set with compact closure contained in $\mathcal{O}$. Then for sufficiently large constants K, γ and $1/\rho$ depending only on $L(x,D)$, $\varphi(x)$ and U:

<u>If</u> $\quad v(x) \equiv \sum_{k=0}^{\infty} v_k(x) f_{k-\ell}[\varphi(x)],$

where each $v_k \in \mathcal{A}(U)$ satisfies

$$|D_o^q D^\beta v_k(x)| \leq AE(r+k+q, |\beta|)(x)$$

for some A, $r \geq 0$ (independent of k) and all $x \in U$, $q \in \mathbb{N}$

$\beta \in \mathbb{N}^n$,

<u>Then</u> $$L(x; D)v(x) \equiv \sum_{k=0}^{\infty} w_k(x) f_{k-\ell-j}[\varphi(x)],$$

where each $w_k \in \mathcal{A}(U)$ and satisfies

$$|D_o^q D^\beta w_k(x)| \leq AKE(r+k+q, |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

<u>Proof:</u> A calculation using Lagrange's formula shows that if $\alpha \in \mathbb{N}^{n+1}$, $|\alpha| \leq j$ and $a_\alpha(x) \in \mathcal{A}(\mathcal{O})$, then

$$a_\alpha(x) D^\alpha (v_k(x) f_{k-\ell}[\varphi(x)]) \equiv \sum_{i=0}^{|\alpha|} a_\alpha(x) L_{i,\alpha}[v_k(x)] f_{k-\ell-|\alpha|+i}[\varphi(x)],$$

where each $L_{i,\alpha}$ is an operator on $\mathcal{O}$ of order $\leq i$ which depends only on φ. Applying Lemma 12.3 to the operators $a_\alpha(x) L_{i,\alpha}(x; D)$, we get

$$\text{(13-10)} \qquad a_\alpha(x) D^\alpha (v_k(x) f_{k-\ell}[\varphi(x)])$$

$$\equiv \sum_{i=0}^{|\alpha|} w^\alpha_{k+(j-|\alpha|)+i}(x) f_{k+(j-|\alpha|)+i-\ell-j}[\varphi(x)],$$

where each $w^\alpha_{k+(j-|\alpha|)+i} \in \mathcal{A}(U)$ and satisfies

(13-11) $$|D_o^q D^\beta w^\alpha_{k+(j-|\alpha|)+i}(x)| \le AK_{i,\alpha} E(r+k+i, |\beta|)(x)$$

for some constants $K_{i,\alpha}$ depending only on a_α and $L_{i,\alpha}$; and for all $x \in U$, $q \in \mathbb{N}$, $\beta \in \mathbb{N}^n$.

Since $j \ge |\alpha| \ge i$, and $E(r, s)(x)$ is an increasing function of r, (13-11) yields

$$|D_o^q D^\beta w^\alpha_{k+(j-|\alpha|)+i}(x)| \le AK_{i,\alpha} E(r+k+(j-|\alpha|)+i+q, |\beta|)(x).$$

Choosing $K_\alpha \ge \sum_{i=1}^{|\alpha|} K_{i,\alpha}$, this and (13-10) give

$$a_\alpha(x) D^\alpha[v(x)] \equiv \sum_{k=0}^{\infty} w_k^\alpha(x) f_{k-\ell-j}[\varphi(x)]$$

where each w_k^α satisfies

$$|D_o^q D^\beta w_k^\alpha(x)| \le AK_\alpha E(r+k+q, |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

Now write $L(x, D) \equiv \sum_{|\alpha| \le j} a_\alpha(x) D^\alpha$. Then $w_k(x) \equiv \sum_\alpha w_k^\alpha(x)$, and letting $K = \sum K_\alpha$ easily gives the required bound on w_k. ∎

We now find bounds for the solutions of the Cauchy problems (10-11).

Lemma 13.4: Let $h_1(x;p), \ldots, h_s(x;p)$ be as in Section 9, and let $b(x;D)$ be an operator on $\mathcal{O}$ of order $\leq m-1$ (where $m = rs$). There exists a neighborhood U of 0 and constants B, C and K such that for all sufficiently large γ and $1/\rho$:

If: $v(x) \equiv w(x) f_{-\ell+(s-1)}[\varphi^{(1)}(x)]$ where $w(x) \in \mathcal{A}(\mathcal{O})$ and $A \geq \sup_{x \in \mathcal{O}} |w(x)|$,

Then: The sequence of Cauchy problems (10-11) possesses formal solutions

$$(13\text{-}12) \qquad {}_j u(x) \equiv \sum_{i=1}^{r} \sum_{k=0}^{\infty} {}_j u_k^{(i)}(x) f_{k-\ell-j(s-1)}[\varphi^{(i)}(x)],$$

where each ${}_j u_k^{(i)} \in \mathcal{A}(U)$ and satisfies

$$(13\text{-}13) \qquad |D_o^q D^\beta {}_j u_k^{(i)}(x)| \leq \frac{KAB^j C^k}{(js)!} E(k+q, |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$.

The choice of U, B, C, K, γ and ρ depends only on $b(x;D)$ and the h_i.

Proof. It suffices to show that for the Cauchy problems (10-11), we can find formal solutions ${}_j u(x)$ of the form (13-12) such that for any $d \geq j$,

$$|D_o^q D^\beta {}_j u_k^{(i)}(x)| \leq \frac{KAB^j C^k}{(ds)!} E((d-j)s+k+q, |\beta|)(x) \tag{13-14}$$

for all $x \in U$, $q \in \mathbb{N}$, $\beta \in \mathbb{N}^n$. Indeed, (13-13) is just (13-14) for $j = d$.

The proof is by induction on j. Assume that U has compact closure in $\mathcal{O}$. Then by Lemma 12.1, for sufficiently large $1/\rho$ depending only on U, and for $\gamma \geq 1/n$, we have

$$|D_o^q D^\beta w(x)| \leq AE(q, |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$, $\beta \in \mathbb{N}^n$. Setting $v_o^{(1)} \equiv w$, and all other $v_k^{(i)} \equiv 0$, we see that $v(x)$ satisfies the hypotheses of Lemma 13.2, with $r = 0$. (Recall that $E(q, |\beta|)(x) \leq E(s+q, |\beta|)(x)$.) Application of Lemma 13.2 gives us a choice of U and K such that for sufficiently large C, γ and $1/\rho$, we have the formal solution ${}_o u(x)$ satisfying

$$|D_o^q D^\beta {}_o u_k^{(i)}(x)| \leq KAC^k E(k+q, |\beta|)(x)$$

for all $x \in U$, $q \in \mathbb{N}$, $\beta \in \mathbb{N}^n$. From (11-1) and (11-5), it is easy to see that

$$E(k+q, |\beta|)(x) \leq \frac{1}{(ds)!} E(ds+k+q, |\beta|)(x).$$

Combining these two inequalities yields (13-14) for the case $j = 0$.

Now assume that we have constructed the solution ${}_j u(x)$ satisfying (13-14) for all $d \geq j$, and let $d \geq j+1$. By (13-12) and (13-14), Lemma 13.3 gives a constant M depending only on $b(x; D)$ and the $\varphi^{(i)}(x)$ such that

$$b(x; D)[{}_j u(x)] = \sum_{i=1}^{r} \sum_{k=0}^{\infty} w_k^{(i)}(x) f_{k-\ell-j(s-1)-(m-1)}[\varphi^{(i)}(x)]$$

and each $w_k^{(i)}$ satisfies

$$|D_o^q D^\beta w_k^{(i)}(x)| \leq \frac{KAMB^j C^k}{(ds)!} E((d-j)s+k+q, |\beta|)(x).$$

Applying Lemma 13.2 for $v(x) \equiv b(x; D)[{}_j u(x)]$, we get a formal solution ${}_{(j+1)}u(x)$ of the form (13-12), since $k-\ell-j(s-1)-(m-1)+m-s = k-\ell-(j+1)(s-1)$, satisfying

$$|D_o^q D^\beta {}_{(j+1)}u_k^{(i)}(x)| \leq \frac{KAM\overline{B}\, B^j C^k}{(ds)!} E([d-(j+1)]s+k+q, |\beta|)(x),$$

for some constant $\bar{B}$ and all $x \in U$, $q \in \mathbb{N}$, $\beta \in \mathbb{N}^n$. Choosing $B \geq M\bar{B}$, this gives (13-14) for $(j+1)$, finishing the induction argument. ∎

Having found bounds on the terms of the formal sums ${}_j u(x)$, we now prove a result which will yield the convergence of the formal solution $\Sigma\, {}_j u(x)$ of our original (restricted) Cauchy problem (10-1).

Lemma 13.5: Assu[redacted]tion g on $\mathbb{C} - \{0\}$, $|f_k(t)| \leq \frac{1}{k!}(2|t|)^k$[redacted] all integers k. With the notations of Le[redacted]ts J, L and M independent of w(x[redacted]ach $i = 1, \ldots, r$:

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |{}_j u_k \text{[redacted]} (x)]|$$

$$\leq g(x) \frac{KA\ell!}{1-M|\varphi^{(i)}(x)|} \exp\left[\frac{L}{|\varphi^{(i)}(x)|^{s-1}}\right]\left[\left(\frac{J}{1-M|\varphi^{(i)}(x)|}\right)^{\ell} + \left(\frac{1}{|\varphi^{(i)}(x)|}\right)^{\ell}\right]$$

for all $x \in U$ such that $0 < |\varphi^{(i)}(x)| < 1/M$.

Proof: Denote $|\varphi^{(i)}(x)|$ by φ. By the assumption on the f_k, it suffices to find J, L and M such that

$$(13\text{-}15) \qquad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |{}_j u_k^{(i)}(x)| \frac{(2\varphi)^{k-\ell-j(s-1)}}{(k-\ell-j(s-1))!}$$

$$\leq \frac{KA\ell!}{1-M\varphi} \exp\left[\frac{L}{\varphi^{s-1}}\right]\left[\left(\frac{J}{1-M\varphi}\right)^{\ell} + \left(\frac{1}{\varphi}\right)^{\ell}\right]$$

whenever $0 < \varphi < 1/M$.

Choose T such that $|t| < T$ for all $(t, y) \in U$. By Lemma 13.4,

$$|{}_j u_k^{(i)}(x)| \leq \frac{KAB^j C^k}{(ds)!} E(k, 0)(x).$$

By (11-3), $E(k, 0)(x) \leq k!\,(E_T)^k$. The above inequality thus implies that the left hand side of (13-15) is less than or equal to

$$(13\text{-}16) \qquad \frac{KA}{(2\varphi)^{\ell}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{k!}{(k-\ell-j(s-1))!\,(js)!} (2E_T C\varphi)^k \left(\frac{B}{\varphi^{s-1}}\right)^j .$$

For convenience, we let X denote $2E_T C\varphi$ and Y denote B/φ^{s-1}. To bound the expression (13-16), we split the sum on k into two parts. First, we have for some constant $\bar{B}$ and all $x \in U$, $q \in \mathbb{N}$, $\beta \in \mathbb{N}^n$. Choosing $B \geq M\bar{B}$, this gives (13-14) for (j+1), finishing the induction argument. ∎

Having found bounds on the terms of the formal sums ${}_j u(x)$, we now prove a result which will yield the convergence of the formal solution $\Sigma\, {}_j u(x)$ of our original (restricted) Cauchy problem (10-1).

Lemma 13.5: Assume that for some function g on $\mathbb{C} - \{0\}$, $|f_k(t)| \leq \frac{1}{k!}(2|t|)^k g(t)$ for all $t \in \mathbb{C} - \{0\}$ and all integers k. With the notations of Lemma 13.4, there exist constants J, L and M independent of $w(x)$ and $\ell \geq 0$, such that for each $i = 1, \ldots, r$:

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| {}_j u_k^{(i)}(x) f_{k-\ell-j(s-1)}[\varphi^{(i)}(x)] \right|$$

$$\leq g(x) \frac{KA\,\ell\,!}{1-M|\varphi^{(i)}(x)|} \exp\left[\frac{L}{|\varphi^{(i)}(x)|^{s-1}}\right]\left[\left(\frac{J}{1-M|\varphi^{(i)}(x)|}\right)^{\ell} + \left(\frac{1}{|\varphi^{(i)}(x)|}\right)^{\ell}\right]$$

for all $x \in U$ such that $0 < |\varphi^{(i)}(x)| < 1/M$.

Proof: Denote $|\varphi^{(i)}(x)|$ by φ. By the assumption on the f_k, it suffices to find J, L and M such that

$$(13\text{-}15) \quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |{}_j u_k^{(i)}(x)| \frac{(2\varphi)^{k-\ell-j(s-1)}}{(k-\ell-j(s-1))!}$$

$$\leq \frac{KA\ell!}{1-M\varphi} \exp\left[\frac{L}{\varphi^{s-1}}\right] \left[\left(\frac{J}{1-M\varphi}\right)^{\ell} + \left(\frac{1}{\varphi}\right)^{\ell}\right]$$

whenever $0 < \varphi < 1/M$.

Choose T such that $|t| < T$ for all $(t, y) \in U$. By Lemma 13.4,

$$|{}_j u_k^{(i)}(x)| \leq \frac{KAB^j C^k}{(ds)!} E(k, 0)(x).$$

By (11-3), $E(k, 0)(x) \leq k!\,(E_T)^k$. The above inequality thus implies that the left hand side of (13-15) is less than or equal to

$$(13\text{-}16) \quad \frac{KA}{(2\varphi)^{\ell}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{k!}{(k-\ell-j(s-1))!(js)!} (2E_T C\varphi)^k \left(\frac{B}{\varphi^{s-1}}\right)^j .$$

For convenience, we let X denote $2E_T C\varphi$ and Y denote B/φ^{s-1}. To bound the expression (13-16), we split the sum on k into two parts. First, we have

$$\sum_{j=0}^{\infty} \sum_{k=j(s-1)+\ell}^{\infty} \frac{k!}{(k-\ell-j(s-1))!(js)!} X^k Y^j$$

$$\leq \sum_{j=0}^{\infty} \sum_{k=j(s-1)+\ell}^{\infty} \frac{k!}{[k-\ell-j(s-1)]![j(s-1)]!j!} X^k Y^j$$

[by (11-5), since $j(s-1) + j = js$]

$$= \sum_{j=0}^{\infty} \sum_{k=j(s-1)+\ell}^{\infty} \frac{(k-\ell)!}{[k-\ell-j(s-1)]![j(s-1)]!} \cdot \frac{k!}{(k-\ell)!} X^k \frac{Y^j}{j!}$$

$$\leq \sum_{j=0}^{\infty} \sum_{k=j(s-1)+\ell}^{\infty} 2^{k-\ell} \frac{k!}{(k-\ell)!} X^k \frac{Y^j}{j!}$$

[by (11-6)]

$$\leq \sum_{j=0}^{\infty} \frac{1}{2^\ell} \ell! \frac{(2X)^\ell}{(1-2X)^{\ell+1}} \frac{Y^j}{j!}$$

if $0 \leq 2X < 1$ [by (11-7)]

$$= \ell! \frac{X^\ell}{(1-2X)^{\ell+1}} \exp(Y).$$

As the second part, we have

$$\sum_{j=0}^{\infty} \sum_{k=0}^{j(s-1)+\ell-1} \frac{k!}{[k-\ell-j(s-1)]!(js)!} X^k Y^j$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j(s-1)+\ell-1} \frac{[\ell+j(s-1)-k]!k!}{(js)!} \cdot X^k Y^j$$

[by definition of $k!$ for $k < 0$]

$$\leq \sum_{j=0}^{\infty} \sum_{k=0}^{j(s-1)+\ell-1} \frac{[\ell+j(s-1)]!}{(js)!} X^k Y^j$$

[by (11-5)]

$$\leq \sum_{j=0}^{\infty} \sum_{k=0}^{j(s-1)+\ell-1} \frac{\ell![\ell+j(s-1)]!}{\ell![j(s-1)]!} \cdot X^k \frac{Y^j}{j!}$$

[by (11-5), since $js = j(s-1) + j$]

$$\leq \sum_{j=0}^{\infty} \sum_{k=0}^{j(s-1)+\ell-1} \ell!\ 2^{[\ell+j(s-1)]} X^k \frac{Y^j}{j!}$$

[by (11-6)]

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j(s-1)+\ell-1} 2^{\ell}\ \ell!\ X^k \frac{(2^{s-1}Y)^j}{j!}$$

$$\leq 2^{\ell}\ \ell! \cdot \frac{1}{1-X} \cdot \sum_{j=0}^{\infty} \frac{(2^{s-1}Y)^j}{j!}, \quad \text{if } 0 \leq x < 1$$

[since $\Sigma\ x^k \leq 1/(1-x)$]

$$= \frac{2^{\ell}\ell!}{1-X} \exp(2^{s-1}Y).$$

Combining these two results and substituting for X and Y, we see that the expression (13-16) is less than or equal to

$$\frac{KA}{(2\varphi)^{\ell}} \ell! \left[\frac{(2E_T C\varphi)^{\ell}}{(1-4E_T C\varphi)^{\ell+1}} \exp[\frac{B}{\varphi^{s-1}}] + \frac{2^{\ell}}{1-2E_T C\varphi} \exp\left[\frac{2^{s-1}B}{\varphi^{s-1}}\right]\right]$$

if $0 \leq 4E_T C\varphi < 1$ and $\varphi > 0$. Setting $J = E_T C$, $M = 4E_T C$ and $L = 2^{s-1}B$, the above expression becomes

$$KA\ \ell! \left[\frac{J^{\ell}}{(1-M\varphi)^{\ell+1}} \exp\left[\frac{2^{1-s}L}{\varphi^{s-1}}\right] + \frac{1}{(1-(M\varphi/2))} \frac{1}{\varphi^{\ell}} \exp\left[\frac{L}{\varphi^{s-1}}\right]\right],$$

and the conditions become $0 < \varphi < 1/M$. Since $2^{1-s}L \leq L$ and $(1-M\varphi) \leq (1-(M\varphi/2))$, it is easy to see that the above expression is $\leq$ the right hand side of (13-15), completing the proof. ∎

14. Proof of Theorem I

In this section, we will prove Theorem I - assuming the existence of the characteristic functions $\varphi^{(i)}$. The proof of their existence is deferred until Chapter IV, since it requires some results which will also be used in proving Theorems II and III.

We first define the functions we will use as the f_k.

Definition 14.1: (1) For $0 < \sigma < 1$,

$$f_k^\sigma(t) \equiv \begin{cases} (-1)^{k+1}\sigma(1-\sigma)\ \ldots\ (|k|-1-\sigma)t^{k+\sigma} & \text{if } k < 0 \\ t^\sigma & \text{if } k = 0 \\ \dfrac{t^{k+\sigma}}{(1+\sigma)\ldots(k+\sigma)} & \text{if } k > 0 \end{cases}$$

(2)

$$f_k^0(t) \equiv \begin{cases} (-1)^{k+1} \dfrac{t^k}{(k+1)!} & \text{if } k < 0 \\ \log t & \text{if } k = 0 \\ \dfrac{1}{k!}\,[t^k \log(t) - A_k t^k] & \text{if } k > 0 \end{cases}$$

where $A_k = (1 + \frac{1}{2} + \ldots + \frac{1}{k})$

The following lemma states that these functions satisfy the assumptions we have made about the f_k.

Lemma 14.2: If $0 \leq \sigma < 1$, then for each integer k:

(1) $\frac{d}{dt} f_k^\sigma(t) \equiv f_{k-1}^\sigma(t)$

(2) $|f_k^\sigma(t)| \leq \frac{1}{k!}(2|t|)^k g^\sigma(t)$ for all $t \in C-\{0\}$,

where

$$g^0(t) \equiv 1 + |\log(t)|$$

$$g^\sigma(t) \equiv |t|^\sigma, \qquad 0 < \sigma < 1.$$

Proof: (1) is easily verified by direct computation. For $\sigma = 0$, (2) follows easily from (11-8) for $k < 0$, and from the fact that $A_k < 2^k$ for $k > 0$. For $\sigma > 0$, the result is also easily obtained, remembering that $1/k! = |k|!$ for $k < 0$. ∎

In order to apply the lemmas of Section 13, we will use the following result:

<u>Lemma 14.3</u>: Let $v(x) \in \mathcal{A}_q(\mathcal{O} - K^{(1)})$. There exists a neighborhood U of 0 contained in $\mathcal{O}$ such that:

(1) There exist functions $w^j_\ell(x)$, $w(x) \in \mathcal{A}(U)$ such that

$$v(x) \equiv \sum_{j=0}^{q-1} \sum_{\ell=0}^{\infty} w^j_\ell(x) f^{j/q}_{-\ell+(s-1)} [\varphi^{(1)}(x)] + w(x),$$

and for any sufficiently small $\varepsilon > 0$, there exists a constant N_ε such that for each w^j_ℓ,

$$\sup_{x \in U} |w^j_\ell(x)| \leq N_\varepsilon \frac{\varepsilon^\ell}{(\ell - s - 1)!} .$$

(2) If $v(x)$ has a polar singularity on $K^{(1)}$, then there exist functions $w^j_k(x)$, $w(x) \in \mathcal{A}(U)$ and $\ell \geq 0$ such that

$$v(x) \equiv \sum_{j=0}^{q-1} \sum_{k=0}^{\ell} w^j_k(x) f^{j/q}_{k-\ell} [\varphi^{(1)}(x)] + w(x),$$

and a constant A such that each w^j_k satisfies

$$|D^i_o D^\beta w^j_k(x)| \leq A\, E(k+i, |\beta|)(x)$$

for all $x \in U$, $i \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$, if γ and ρ are sufficiently large.

(3) The conclusions of (1) and (2) remain valid if $h(x)\log[\varphi^{(1)}(x)]$ is added to $v(x)$, for any $h(x) \in \mathcal{A}(\mathcal{O})$.

(4) If $v(x)$ is algebraic on $\mathcal{O}$, then there exist functions $w_j(x)$, $w(x) \in \mathcal{A}(\mathcal{O})$ such that

$$v(x) \equiv \sum_{j=1}^{q-1} w_j(x) f_o^{j/q}[\varphi^{(1)}(x)] + w(x).$$

Proof: By Proposition 3.6, we can find neighborhoods M of 0 in $\mathbb{C}$ and N of 0 in $\mathbb{C}^{n+1}$ contained in $\mathcal{O}$, and a function $h(t, x) \in \mathcal{A}((M-\{0\}) \times N)$ such that $v(x) \equiv h([\varphi^{(1)}(x)]^{1/q}, x)$ for $x \in N-K^{(1)}$.

The Laurent expansion of $h(t, x)$ about $t = 0$ gives

(14-1) $$v(x) \equiv \sum_{k=-\infty}^{\infty} v_k(x)([\varphi^{(1)}(x)]^{1/q})^k,$$

where $v_k(x) \in \mathcal{A}(N)$ and the sum is absolutely convergent for all $x \in N$ with $|\varphi^{(1)}(x)|$ sufficiently small. Moreover, the sum

(14-2) $$\sum_{k=-\infty}^{\infty} v_k(x)\, \epsilon^{k/q}$$

is absolutely convergent on N for sufficiently small $\epsilon > 0$. Letting

$$v_{\ell}^{j}(x) \equiv \begin{cases} v_{[-\ell+(s-1)]q+j}(x) & \text{if } \ell > s-1 \\ 0 & \text{if } \ell = s-1, \ j = 0 \\ \sum_{k=0}^{\infty} v_{kq+j}(x)[\varphi^{(1)}(x)]^{k} & \text{if } \ell = s-1, \ j = 1, \ldots, q-1, \end{cases}$$

$$w(x) \equiv \sum_{k=0} v_{kq}(x)[\varphi^{(1)}(x)]^{k},$$

(14-1) gives

$$(14\text{-}3) \qquad v(x) \equiv \sum_{\ell=s-1}^{\infty} v_{\ell}^{j}(x)[\varphi^{(1)}(x)]^{(-\ell+(s-1)+j/q)} + w(x),$$

where $v_{\ell}^{j} \in \mathcal{A}(N)$ and the sum is absolutely convergent for $|\varphi^{(1)}(x)|$ sufficiently small. Choose U to be an open neighborhood of 0 with compact closure contained in N such that $|\varphi^{(1)}(x)|$ is small enough to ensure absolute convergence of (14-3) for all $x \in U$.

By the absolute convergence of (14-2), for sufficiently small $\varepsilon > 0$, we can choose a constant M_{ε} such that

$$M_{\varepsilon} \geq \sup_{x \in U} |v_{\ell}^{j}(x)| \, \varepsilon^{(-\ell+(s-1)+j/q)}$$

for all j, ℓ. Then if $\varepsilon < 1$,

$$(14\text{-}4) \qquad \sup_{x\in U} |v_\ell^j(x)| \le \frac{M_\varepsilon}{\varepsilon^{(s-1+(q-1)/q)}} \cdot \varepsilon^\ell .$$

Setting

$$v_\ell^j(x)[\varphi^{(1)}(x)]^{(-\ell+(s-1)+j/q)} \equiv w_\ell^j(x) f^j_{-\ell+(s-1)}[\varphi^{(1)}(x)],$$

Definition 14.1 gives

$$(14\text{-}5) \quad (a) \qquad w_\ell^o(x) \equiv (-\ell+(s-1)+1)!\, v_\ell^o(x) \equiv \frac{v_\ell^o(x)}{(\ell-s)!}$$

$$(b) \qquad w_\ell^j(x) \equiv \frac{(-1)^{(-\ell+s)} v_\ell^j(x)}{(j/q)[1-(j/q)]\dots[\ell-s-(j/q)]}, \quad j = 1,\dots,q-1,$$

so (14-5) and (14-3) yield the required sum representation of $v(x)$ for (1).

To find the required N_ε, we observe that

$$(\ell-s)! \ge (\ell-s-1)!$$

$$(j/q)[1-(j/q)] \dots [\ell-s-(j/q)] \ge (1/q)[1-(q-1)/q](\ell-s-1)!$$

for $0 < j < q$. Then setting

$$N_\varepsilon = \frac{M_\varepsilon}{\varepsilon^{(s-1+(q-1)/q)}(1/q)(1-(q-1)/q)},$$

(14-4) and (14-5) yield

$$\sup_{x\in U} |w_\ell^j(x)| \leq N_\varepsilon \frac{\varepsilon^\ell}{(\ell-s-1)!} .$$

This proves part (1).

Observe that adding $h(x)\log[\varphi^{(1)}(x)]$ just requires setting $w_{s-1}^o(x) \equiv h(x)$. Choosing N_ε so that it also satisfies

$$\sup_{x\in U} |w_{s-1}^o(x)| \leq N_\varepsilon \frac{\varepsilon^{s-1}}{(-2)!}$$

is trivial. This proves (3) for part (1).

For Part (2), we note that if $v(x)$ has a polar singularity, then the sum in (14-1) can be taken from $k = -q\ell$ to ∞ for some ℓ. Setting

$$w_k^o(x) \equiv (k-\ell+1)!\; v_{q(k-\ell)}(x), \qquad 0 \leq k < \ell$$

$$w_\ell^o(x) \equiv 0$$

$$w_k^j(x) \equiv (-1)^{k-\ell+1} v_{q(k-\ell)+j}(x)/(j/q)[1-(j/q)] \ldots [\ell-k-1-(j/q)]$$

$$\text{for} \quad 0 \leq k < \ell\,, \quad j = 1, \ldots, q-1$$

$$w_{\ell}^{j}(x) \equiv \sum_{k=0}^{\infty} v_{kq+j}(x)[\varphi^{(1)}(x)]^{k}, \qquad j = 1, \ldots, q-1$$

$$w(x) \equiv \sum_{k=0}^{\infty} v_{kq}(x)[\varphi^{(1)}(x)]^{k},$$

gives the required sum representation of $v(x)$.

By Lemma 12.1, and the definition of $E(i, j)(x)$, if

$$A \geq \sup_{x \in U} |w_{k}^{j}(x)|$$

for each $w_{k}^{j}(x)$, then

$$|D_{o}^{i} D^{\beta} w_{k}^{j}(x)| \leq A\, E(i, |\beta|)(x)$$

for all $x \in U$, $i \in \mathbb{N}$, $\beta \in \mathbb{N}^{n}$. Since $E(i, |\beta|)(x) \leq E(i+k, |\beta|)(x)$ for $k \geq 0$, this proves part (2).

Adding $h(x) \log[\varphi^{(1)}(x)]$ to $v(x)$ simply requires setting $w_{\ell}^{o}(x) \equiv h(x)$, which does not change the above argument. This completes the proof of Part (3).

Finally, assume that $v(x)$ is algebraic on U. Then the sum in (14-1) can be taken from $k = 0$ to ∞, so we can take $\ell = 0$ in Part (2). Since $w_{\ell}^{o}(x) \equiv 0$, setting $w_{j}(x) \equiv w_{\ell}^{j}(x)$, Part (2) then implies Part (4). ∎

We now prove the theorem. Replace $\mathcal{O}$ by the neighborhood U of Lemma 14.3. By Part (1) of Lemma 14.3, using the superposition principle (since a(x; D) is linear), we need only prove part 1 (a) of the theorem for

$$v(x) \equiv \sum_{j=0}^{q-1} \sum_{\ell=0}^{\infty} w_{\ell}^{j}(x) f_{-\ell+(s-1)}^{j/q}[\varphi^{(1)}(x)] \tag{14-6}$$

for each j, where the w_{ℓ}^{j} are as in the lemma. (Since w(x) is analytic, the solution for $v(x) \equiv w(x)$ follows from the Cauchy-Kowalewski Theorem.)

Let ${}_{d}^{j}u_{k}^{(i)}(x)$ be the functions found in Lemma 13.4 for $w(x) \equiv w_{\ell}^{j}(x)$, and let

$$u_{\ell}^{(i)}(x) \equiv \sum_{j=0}^{q-1} \sum_{d=0}^{\infty} \sum_{k=0}^{\infty} {}_{d}^{j}u_{k}^{(i)}(x).$$

Then $u_{\ell}(x) \equiv \sum_{i=1}^{r} u_{\ell}^{(i)}(x)$ is a formal solution to the (reduced) original Cauchy problem (10-1) for $v(x) \equiv \sum_{j=1}^{q-1} w_{\ell}^{j}(x) f_{-\ell+(s-1)}^{j/q}[\varphi^{(1)}(x)]$. Lemma 13.5, together with Lemmas 14.2 (2) and 14.3 (2) show that the formal sum $u_{\ell}^{(i)}(x)$ converges on a neighborhood U of 0 which is independent of ℓ, so $u_{\ell}(x)$ is a function which solves the above Cauchy problem. Moreover, it is easy to see that

$$u_{\ell}^{(i)}(x) \equiv \sum_{j=1}^{q-1} F_{j}^{(i)}(x) + F^{(i)}(x) + G^{(i)}(x)\log[\varphi^{(i)}(x)],$$

where $F_j^{(i)}(x) \in \mathcal{A}_q(U-K^{(i)})$, $F^{(i)}(x) \in \mathcal{A}(U-K^{(i)})$ and $G^{(i)}(x) \in \mathcal{A}(U)$. Hence, $u_\ell(x)$ is a function of the form (6-1).

Using the bound on $\sup_{x \in U} |w_\ell^j(x)|$ of Lemma 14.3 in Lemma 13.5, we get

$$(14\text{-}7) \qquad |u_\ell^{(i)}(x)| \leq \sum_{j=0}^{q-1} \left\{ \frac{g^{j/q}(x) K N_\epsilon}{1-M|\varphi^{(i)}(x)|} \exp\left[\frac{L}{|\varphi^{(i)}(x)|^{s-1}}\right] \cdot \frac{\ell!}{(\ell-s-1)!} \epsilon^\ell \left[\left(\frac{J}{1-M|\varphi^{(i)}(x)|}\right)^\ell + \left(\frac{1}{|\varphi^{(i)}(x)|}\right)^\ell\right]\right\}$$

for all $x \in U$ with $0 < |\varphi^{(i)}(x)| < 1/M$, and any sufficiently small ϵ, where K, L, M and N_ϵ are constants which are independent of ℓ.

Let U be small enough so that $|\varphi^{(i)}(x)| < 1/M$ for all $x \in U$, $i = 1, \ldots, r$. Then (14-7) holds for all $x \in U-K^{(i)}$.

Now let $u^{(i)}(x) \equiv \sum_{\ell=0}^{\infty} u_\ell^{(i)}(x)$, and $u(x) \equiv \sum_{i=1}^{r} u^{(i)}(x)$. Then the superposition principle implies that $u(x)$ solves the Cauchy problem (10-1) for the $v(x)$ given by (14-6), if the sum defining each $u^{(i)}(x)$ is absolutely convergent.

Given any $x > 0$, there is a sufficiently small $\epsilon > 0$ such that

$$\sum_{\ell=0}^{\infty} \frac{\ell!}{(\ell-s-1)!} \epsilon^\ell x^\ell$$

converges. Using this, it is easy to see from (14-7) that $\sum_{\ell=0}^{\infty} |u_\ell^{(i)}(x)|$ converges for all $x \in U-K^{(i)}$. Since each $u_\ell(x)$ is of the form (6-1), $u(x)$ is also. Hence, $u(x)$ is the required solution function of part 1 (a) of the theorem.

The uniqueness of $u(x)$ follows easily from the uniqueness part of the Cauchy Kowalewski Theorem, since $a(x; D)u(x) \equiv a(x; D)\bar{u}(x)$ implies $a(x; D)[u(x)-\bar{u}(x)] \equiv 0$. This completes the proof of part 1 (a).

For part 1 (b) of Theorem I, we combine Lemma 14.3 (2) and Lemma 13.1 to get the neighborhood U and formal solution

$$(14\text{-}8) \qquad u(x) \equiv \sum_{j=0}^{q-1} \sum_{i=1}^{r} \sum_{k=0}^{\infty} {}_j u_k^{(i)}(x) f_{k-\ell+r-1}^{j/q}[\varphi^{(i)}(x)]$$

with

$$(14\text{-}9) \qquad |{}_j u_k^{(i)}(x)| \leq ABC^k E(r+k, 0)(x),$$

for all $x \in U$, where B and C are constants independent of k. Then (14-9), (11-4) and Lemma 14.2 (2) yield

$$|{}_j u_k^{(i)}(x) f_{k-\ell+r-1}^{j/q}(x)|$$

$$\leq AB\, g^{j/q}(x)(E_T)^r (2|\varphi^{(i)}(x)|)^{-\ell+r-1} \cdot \frac{(r+k)!}{(r+k-\ell-1)!}[2C|\varphi^{(i)}(x)|]^k.$$

This implies that the sum on k in (14-8) converges absolutely on $U-K^{(i)}$, if U is small enough so that $|\varphi^{(i)}(x)| < 1/2C$ for all $x \in U$. Using (14-8), it is easy to write $u(x)$ in the form (6-1), where the $F^{(i)}$ have only polar singularities along $K^{(i)}$. This proves part 1 (b).

An examination of the proof of part 1 of the theorem shows that Remark 6.1 follows from Lemma 14.3 (3).

For part 2 (a), observe that the logarithmic terms in the solution $u(x)$ come from the functions $f_k^o[\varphi^{(i)}(x)]$. If these functions are missing from the expression (14-8) for $v(x)$, then they will not appear in $u(x)$. Hence, we will have each $G^i(x) \equiv 0$ if $v(x)$ can be represented in the form (14-8) with the sum on j running from 1 to $q-1$. By Lemmas 14.3 (4) and 13.3, this is the case if $v(x) \equiv L(x; D)V(x)$ for any algebraic function $V(x)$ on $\mathcal{O}$ with singularities along $K^{(1)}$, and any operator $b(x; D)$. This proves part 2 (a).

Finally, observe that for $j > 0$, the sum on k in (14-8) is an algebraic function with singularities on $K^{(i)}$, if $\ell \leq r-1$. By Lemma 14.3 (4) and Lemmas 10.1 and 13.3, this is the case if $v(x)$ is algebraic, or $v(x) \equiv a(x; D)V(x)$ for any algebraic function $V(x) \in \mathcal{A}_q(U-K^{(1)})$. This proves part 2 (b), completing our proof of Theorem I (assuming the existence of the $\varphi^{(i)}$). ∎

III. PROOF OF THEOREM IV

15. Proof of Proposition 8.5

In this section, we prove Proposition 8.5. We begin by proving an elementary result which will be needed. Let $|\cdot|$ denote the usual norm on $\mathbb{C}^2$, defined by $|(t,y)|^2 = |t|^2 + |y|^2$.

Lemma 15.1: Let $\beta : [\mu, \rho] \to \mathbb{C}^2$ be a differentiable real path satisfying

$$\left|\frac{d\beta}{d\tau}(\tau)\right| \leq L\ |\beta(\tau)|$$

for some constant L, and all $\tau \in [\mu, \rho]$. Then

$$|\beta(\tau)| \leq \exp(L\ |\tau_0 - \tau|)\ |\beta(\tau_0)|$$

for any $\tau, \tau_0 \in [\mu, \rho]$.

Proof: Assume $\tau \geq \tau_0$. We then have

$$|\beta(\tau) - \beta(\tau_0)| = \left|\int_{\tau_0}^{\tau} \frac{d\beta}{d\sigma}(\sigma) d\sigma\right|$$

$$\leq \int_{\tau_0}^{\tau} \left|\frac{d\beta}{d\sigma}(\sigma)\right| d\sigma$$

$$\leq \int_{\tau_0}^{\tau} L\,|\beta(\sigma)|\,d\sigma\ .$$

Since $|\beta(\tau) - \beta(\tau_0)| \geq |\beta(\tau)| - |\beta(\tau_0)|$, this gives

$$|\beta(\tau)| \leq |\beta(\tau_0)| + L\int_{\tau_0}^{\tau} |\beta(\sigma)|\,d\sigma$$

A change of variables then reduces the problem to proving that if f is a continuous real-valued function satisfying

(15-1) $$f(\tau) \leq f(0) + L\int_0^{\tau} f(\sigma)d\sigma$$

for each $\tau \in [0, \lambda]$, then

(15-2) $$f(\tau) \leq f(0)\exp(L\tau).$$

Let $M = \sup_{\tau\in[0,\lambda]} f(\tau)$. We will prove by induction that for every n,

(15-3) $$f(\tau) \leq f(0)\left[\sum_{k=0}^{n} \frac{(L\tau)^k}{k!}\right] + M\frac{(L\tau)^{n+1}}{(n+1)!}$$

for all $\tau \in [0, \rho]$. This inequality then obviously implies (15-2).

For $n = 1$, (15-3) follows easily from (15-1), since $\int_0^\tau f(\sigma)d\sigma \leq M\tau$. Assume that (15-3) holds for n. Applying (13-3) to $f(\sigma)$ in (15-1), we get

$$f(\tau) \leq f(0) + L\int_0^\tau (f(0)[\sum_{k=0}^{n} \frac{(L\sigma)^k}{k!}] + M\frac{(L\sigma)^{n+1}}{(n+1)!})d\sigma .$$

A simple integration shows that this inequality is just (15-3) with $n + 1$ substituted for n.

This completes the proof for the case $\tau \geq \tau_0$. The proof for $\tau < \tau_0$ just requires changing all integrals $\int_{\tau_0}^{\tau} \cdots$ to $\int_{\tau}^{\tau_0} \cdots$. $\blacksquare$

We now recall our notation. N is a star-shaped neighborhood of 0 in C, and $\mathcal{D}$ is a star-like neighborhood of 0 with $\eta : N \to \mathcal{D}$ is a star-like neighborhood of 0 with $\eta : N \to \mathcal{D}$ the mapping of Definition 8.2. The open set $\mathcal{O}$ is a subset of $\mathcal{D} \times C$, and the functions $\alpha^{(i)} \in \mathcal{A}(\mathcal{O})$. For each $x_0 \in \mathcal{O}$, $\gamma_{x_0}^{(i)}$ is the complex path in $\mathcal{O}$ defined by (8-2).

We now prove a result which puts a bound on the change in $\gamma_x(t)$ as x changes. Recall the definitions of η_θ and $\|t\|$ given in Section 11.

<u>Lemma 15.2:</u> Let K be a compact subset of $\mathcal{O}$. There exist constants L and $\varepsilon > 0$ such that for any $x_0, x_1 \in K$ with $|x_1 - x_0| < \varepsilon$,

where $x_0 = (t_0, y_0)$ and $t_0 = \eta_\theta(\|t_0\|)$, and for each $i = 1, \ldots, m$:

$$|\gamma_{x_1}^{(i)}[\eta_\theta(\tau)] - \gamma_{x_0}^{(i)}[\eta_\theta(\tau)]| \tag{15-4}$$

$$\leq \exp[L(\|t_0\| - \tau)] \, |\gamma_{x_1}^{(i)}(t_0) - \gamma_{x_0}^{(i)}(t_0)|$$

for any τ such that $\gamma_{x_0}^{(i)}[\eta_\theta(\tau')], \gamma_{x_1}^{(i)}[\eta_\theta(\tau')] \in K$ for all $\tau' \in [\tau, \|t_0\|]$.

Proof: By the compactness of K and the analyticity of the $\alpha^{(i)}$, we can find constants A and $\delta > 0$ such that

$$|\alpha^{(i)}(z) - \alpha^{(i)}(x)| \leq A|z-x|$$

for all $x, z \in K$ with $|z-x| < \delta$. Since η is continuously differentiable, we can also find a constant B satisfying $|\frac{d\eta_\theta}{d\tau}(\tau)| \leq B$ for all θ, τ such that $(\eta_\theta(\tau), y) \in K$ for some y.

We now compute

$$|\frac{d}{d\tau}(\gamma_{x_1}^{(i)}[\eta_\theta(\tau)] - \gamma_{x_0}^{(i)}[\eta_\theta(\tau)])|$$

$$= |(0, [\alpha^{(i)}(\gamma_{x_0}^{(i)}[\eta_\theta(\tau)]) - \alpha^{(i)}(\gamma_{x_1}^{(i)}[\eta_\theta(\tau)])]\frac{d\eta_\theta}{d\tau}(\tau))|$$

[by (8-2)]

For $n = 1$, (15-3) follows easily from (15-1), since $\int_0^\tau f(\sigma)d\sigma \leq M\tau$. Assume that (15-3) holds for n. Applying (13-3) to $f(\sigma)$ in (15-1), we get

$$f(\tau) \leq f(0) + L\int_0^\tau \left(f(0)\left[\sum_{k=0}^{n} \frac{(L\sigma)^k}{k!}\right] + M\frac{(L\sigma)^{n+1}}{(n+1)!}\right)d\sigma .$$

A simple integration shows that this inequality is just (15-3) with $n + 1$ substituted for n.

This completes the proof for the case $\tau \geq \tau_0$. The proof for $\tau < \tau_0$ just requires changing all integrals $\int_{\tau_0}^{\tau} \cdots$ to $\int_{\tau}^{\tau_0} \cdots$. ∎

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where $x_0 = (t_0, y_0)$ and $t_0 = \eta_\theta(\|t_0\|)$, and for each $i = 1, \ldots, m$:

(15-4)
$$|\gamma^{(i)}_{x_1}[\eta_\theta(\tau)] - \gamma^{(i)}_{x_0}[\eta_\theta(\tau)]|$$

$$\leq \exp[L(\|t_0\| - \tau)]\, |\gamma^{(i)}_{x_1}(t_0) - \gamma^{(i)}_{x_0}(t_0)|$$

for any τ such t[illegible] $(\tau')] \in K$ for all $\tau' \in [\tau, \|t_0\|]$.

Proof: By t[illegible] analyticity of the $\alpha^{(i)}$, we can find const[illegible]

$$[illegible] | \leq A|z-x|$$

for all $x, z \in K$ with $|z-x| < \delta$. Since η is continuously differentiable, we can also find a constant B satisfying $|\frac{d\eta_\theta}{d\tau}(\tau)| \leq B$ for all θ, τ such that $(\eta_\theta(\tau), y) \in K$ for some y.

We now compute

$$|\frac{d}{d\tau}(\gamma^{(i)}_{x_1}[\eta_\theta(\tau)] - \gamma^{(i)}_{x_0}[\eta_\theta(\tau)])|$$

$$= |(0, [\alpha^{(i)}(\gamma^{(i)}_{x_0}[\eta_\theta(\tau)]) - \alpha^{(i)}(\gamma^{(i)}_{x_1}[\eta_\theta(\tau)])]\frac{d\eta_\theta}{d\tau}(\tau))|$$

[by (8-2)]

$$\leq AB\,|\gamma_{x_1}^{(i)}[\eta_\theta(\tau)] - \gamma_{x_0}^{(i)}[\eta_\theta(\tau)]|$$

if $|\gamma_{x_1}^{(i)}[\eta_\theta(\tau)] - \gamma_{x_0}^{(i)}[\eta_\theta(\tau)]| < \delta$.

Applying Lemma 15.1 to the path $\tau \to \gamma_{x_1}^{(i)}[\eta_\theta(\tau)] - \gamma_{x_0}^{(i)}[\eta_\theta(\tau)]$, substituting $\|t_0\|$ for τ_0 and letting $L = AB$, yields (13-14) - if we make the additional assumption that

(15-5) $$|\gamma_{x_1}^{(i)}[\eta_\theta(\tau')] - \gamma_{x_0}^{(i)}[\eta_\theta(\tau')]| < \delta$$

for all $\tau' \in [\tau, \|t_0\|]$.

The compactness of K and the continuity of η^{-1} allow us to choose T such that $T \geq \|t\|$ for all $(t, y) \in K$. By the continuity of the mappings $(t, x) \to \gamma_x^{(i)}(t)$ on a neighborhood of the compact set $\{(t, (t, y)) : (t, y) \in K\}$, we can find an $\epsilon > 0$ such that $|x_1 - x_0| < \epsilon$ implies $\gamma_{x_1}^{(i)}(t_0)$ is defined and $|\gamma_{x_1}^{(i)}(t_0) - \gamma_{x_0}^{(i)}(t_0)| < \delta/\exp(LT)$.

Now, applying (15-4) to the points τ' shows that (15-5) holds so long as $\gamma_{x_1}^{(i)}[\eta_\theta(\tau')]$, $\gamma_{x_0}^{(i)}[\eta_\theta(\tau')] \in K$. More precisely, assume that $\gamma_{x_0}^{(i)}[\eta_\theta(\tau')]$, $\gamma_{x_1}^{(i)}[\eta_\theta(\tau')] \in K$ for all $\tau' \in [\tau, \|t_0\|$, and let $\tau'' = \sup\{\tau' \in [\tau, \|t_0\|]:: \text{(15-4) holds for } \tau'\}$. It is easy to apply what we have already proved (i.e., that (15-4) holds under the additional assumption (15-5) holds for all $\tau' \in [\tau, \|t_0\|]$) to the points $\gamma_{x_0}^{(i)}[\eta_\theta(\tau'')]$, $\gamma_{x_1}^{(i)}[\eta_\theta(\tau'')]$ in order to show that $\tau'' = \tau$. Thus, (15-4) holds for τ . $\blacksquare$

With the aid of the paths η_θ, we can rewrite the definition of the dependence path $\beta^{(i)}_{x_0}$ as

$$\beta^{(i)}_{x_0}(\tau) \equiv \gamma^{(i)}_{x_0}[\eta_\theta(\tau)],$$

where $x_0 = (t_0, y_0)$ and $t_0 = \eta_\theta(\|t_0\|)$. Note that $\beta^{(i)}_{x_0}(\|t_0\|) = x_0$.

We drop the requirement that $\beta^{(i)}_{x_0}$ be defined on the entire interval $[0, \|t_0\|]$. Instead, we just assume that $\beta^{(i)}_{x_0}(\rho)$ is defined when $\gamma^{(i)}_{x_0}[\eta_\theta(\tau)]$ is defined for all $\tau \in [\rho, \|t_0\|]$.

To prove that $\mathcal{J}(V)$ is open, we need a method of proving that a particular point x is in $\mathcal{J}(K)$. To obtain it, we first make the following definition.

<u>Definition 15.3</u>: Let $x = (t, y) \in \mathcal{O}$. A <u>broken dependence path to x</u> is any continuous real path $\beta : [\rho\|t\|, \|t\|] \to \mathcal{O}$ for $0 \le \rho \le 1$ such that for some numbers ρ_j with $\rho = \rho_0 < \rho_1 < \ldots < \rho_n = 1$ and some i_j: if $\tau \in [\rho_{j-1}, \rho_j]$, then $\beta(\tau\|t\|) = \beta^{(i_j)}_{\beta(\rho_j\|t\|)}(\tau\|t\|)$.

We say that β can be <u>extended to 0</u> if $\beta^{(i_1)}_{\beta(\rho_1\|t\|)}$ is defined on $[0, \rho_1\|t\|]$. In this case, we can assume that β is defined on $[0, \|t\|]$.

Thus, a broken dependence path is obtained by piecing together segments of dependence paths $\beta_z^{(i)}$ to form a continuous path. This definition allows us to give the following characterization of $\mathcal{J}(K)$.

Lemma 15.4: Let K be a compact subset of $\mathcal{O}$, and $V \subset \mathcal{O} \cap \mathbb{C}$. Then the point $x = (t, y) \in K$ is an element of $\mathcal{J}(K)$ if and only if every broken dependence path β to x satisfies:

(1) β can be extended to 0, and

(2) $\beta([0, \|t\|]) \subset K$

Proof: Let $\mathcal{J}$ be the set of all elements $x \in K$ with the above property. We must show that $\mathcal{J} = \mathcal{J}(K)$. Recall that $\mathcal{J}(K)$ is defined to be the largest set $S \subset K$ such that for every $x = (t, y) \in S$ and each $i = 1, \ldots, m$:

(1') $\beta_x^{(i)}$ is defined on $[0, \|t\|]$, and

(2') $\beta_x^{(i)}([0, \|t\|]) \subset S$

Since a dependence path $\beta_x^{(i)}$ is a special case of a broken dependence path to x, any point $x \in \mathcal{J}$ obviously satisfies (1') for any i. For any $x \in \mathcal{J}$, let $z = (s, w) \in \beta_x^{(i)}([0, \|t\|])$ and let β be any broken dependence path to z. We can extend β to a broken dependence path $\overline{\beta}$ to x by letting

$$\overline{\beta}(\tau) = \begin{cases} \beta(\tau) & \text{if} \quad \tau \leq \|s\| \\ \beta_x^{(i)}(\tau) & \text{if} \quad \|s\| \leq \tau \leq \|t\| \end{cases} .$$

Since $\overline{\beta}$ satisfies conditions (1) and (2) above, β does also. Thus, $z \in \mathcal{J}$ and (2') is satisfies with $S = \mathcal{J}$. By the maximality of $\mathcal{J}(K)$, this proves that $\mathcal{J} \subset \mathcal{J}(K)$.

Now let $x \in \mathcal{J}(K)$ and let β be any broken dependence path to x. With the notation of Definition 15.3, we can apply (1') and (2') successively to the paths $\beta^{(i_j)}_{\beta(\rho_j\|t\|)}$ for $j = n, n-1, \ldots, 1$. Since each of these paths lies in $\mathcal{J}(K) \subset K$, this shows that β satisfies (2). Since $\beta^{(i_1)}_{\beta(\rho_1\|t\|)}$ satisfies (1'), β satisfies (1). Hence $x \in \mathcal{J}$, proving that $\mathcal{J}(K) \subset \mathcal{J}$. ∎

Given a broken dependence path β to x and a point x_1 close to x, we want to choose a broken dependence path to x_1 which is close to β. We thus make the following definition.

<u>Definition 15.5:</u> Let $x, x_1 \in \mathcal{O}$ and let β be a broken dependence path to x. The broken dependence path to x_1 <u>parallel to</u> β is the path $\overline{\beta}$ defined as follows. Let $x_1 = (t_1, y_1)$ and let the ρ_j, i_j be as in Definition 13.3. Then $\overline{\beta}(\|t_1\|) = x_1$ and for $j = n, n-1, \ldots, 1$: if $\tau \in [\rho_{j-1}, \rho_j]$, then

$$\overline{\beta}(\tau\|t_1\|) = \beta^{(i_j)}_{\overline{\beta}(\rho_j\|t_1\|)}(\tau\|t_1\|),$$

so long as this is defined.

In general, if β is defined on $[\rho\|t\|, \|t\|]$, then $\overline{\beta}$ will be defined on $[\overline{\rho}\|t_1\|, \|t_1\|]$ for some $\overline{\rho} \geq \rho$. Note that if $\overline{\beta}$ is parallel to β, then β is parallel to $\overline{\beta}$. (More precisely, this is true if we restrict the domain of β to $[\overline{\rho}\|t\|, \|t\|]$.)

We now show that if x_1 is close to x, then $\overline{\beta}$ is close to β.

Lemma 15.6: Let K, $\overline{K}$ be compact subsets of $\mathcal{O}$ such that K is contained in the interior of $\overline{K}$. There exists an $\epsilon > 0$ such that for each $i = 1, \ldots, m$:

If $x_0 = (t_0, y_0) \in K$ and β is a broken dependence path to x_0 defined on $[\rho\|t_0\|, \|t_0\|]$ with $\beta([\rho\|t_0\|, \|t_0\|]) \subset K$,

Then for any $x_1 = (t_1, y_1)$ with $|x_1 - x_0| < \epsilon$, the dependence path $\overline{\beta}$ to x_1 parallel to β is defined on $[\rho\|t_1\|, \|t_1\|]$, and $\beta([\rho\|t_1\|, \|t_1\|) \subset \overline{K}$.

Proof: Since the mappings $(t, x) \to \gamma_x^{(i)}(t)$ are defined and analytic on a neighborhood of the compact set $\{(t, (t, y)) : (t, y) \in \overline{K}\}$, we can choose constants A, $\delta_1 > 0$ with the following property: if $x = (t, y) \in \overline{K}$ and $|t'-t| < \delta_1$, then $\gamma_x^{(i)}(t')$ is defined and

$$|\gamma_x^{(i)}(t') - \gamma_x^{(i)}(t)| \leq A\,|t'-t|. \tag{15-6}$$

We can also find constant B, $\delta_2 > 0$ such that $x = (t, y)$, $x' \in \overline{K}$ and

$|x'-x| < \delta_2$ imply $\gamma_{x'}(t)$ is defined and

(15-7) $$|\gamma_{x'}^{(i)}(t) - \gamma_x^{(i)}(t)| \le B\,|x'-x|.$$

Similarly, the differentiability of η on the compact set $S = \{\rho t : (\eta(t), y) \in \overline{K}$ for some y, $0 \le \rho \le 1\}$ allows us to choose constants C, $\delta_3 > 0$ such that for any $t, t' \in S$ with $|t'-t| < \delta_3$ and any $0 \le \tau \le 1$:

$$|\eta(\tau t') - \eta(\tau t)| \le C\tau\,|t'-t|.$$

We can restate this as follows. If $t = \eta_\theta(\|t\|)$, $t' = \eta_\chi(\|t'\|)$, and $|t'-t| < \delta_3$, then

(15-8) $$|\eta_\chi(\tau\|t'\|) - \eta_\theta(\tau\|t\|)| \le C\tau\,|t'-t|$$

whenever $0 \le \tau \le 1$.

Choose constant L, $\overline{\epsilon}$ as in Lemma 15.2 for the set $\overline{K}$. Using the compactness of K, we can find $\delta_4 > 0$ such that $x \in K$ and $|z-x| < \delta_4$ imply that z is contained in the interior of $\overline{K}$. Finally, choose $T \ge \|t\|$ for all $(t, y) \in \overline{K}$.

Now let the ρ_j, i_j be as in Definition 15.5. Let $\widetilde{\beta}$ be the path defined by

$$\tilde{\beta}(\tau\|t_0\|) = \gamma^{(i_j)}_{\bar{\beta}(\rho_j\|t_1\|)} [\eta_\theta(\tau\|t_0\|)],$$

where $t_0 = \eta_\theta(\|t_0\|)$. The path $\tilde{\beta}$ thus "lies between" the paths β and $\bar{\beta}$.

If $\varepsilon \leq \bar{\varepsilon}$, successive applications of Lemma 15.2 for $j = n, n-1, \ldots, 1$ - substituting $\tilde{\beta}(\rho_j\|t_0\|)$ for x_1, $\beta(\rho_j\|t_0\|)$ for x_0 and i_j for i - gives

$$|\tilde{\beta}(\tau\|t_0\|) - \beta(\tau\|t_0\|)| \leq \exp[L(\|t_0\| - \tau\|t_0\|)] \, |\gamma^{(i_n)}_{x_1}(t_0) - \gamma^{(i_n)}_{x_0}(t_0)|$$

so long as $\tilde{\beta}(\tau'\|t_0\|)$, $\beta(\tau'\|t_0\|)$ are defined and in $\bar{K}$ for all $\tau' \in [\tau, 1]$.

If $\varepsilon \leq \delta_2$, then applying (15-7) gives

$$(15\text{-}9) \qquad |\tilde{\beta}(\tau\|t_0\|) - \beta(\tau\|t_0\|)| \leq B \exp(LT) \, |x_1 - x_0|$$

whenever $\tilde{\beta}(\tau'\|t_0\|)$, $\beta(\tau'\|t_0\|) \in \bar{K}$ for all $\tau' \in [\tau, 1]$.

Now choose ε smaller than δ_3 and δ_1/C. Then $|t_1 - t_0| \leq |x_1 - x_0| < \varepsilon$, together with (15-8) implies that

$$(15\text{-}10) \qquad |\eta_\chi(\tau\|t_1\|) - \eta_\theta(\tau\|t_0\|)| < \delta_1$$

for all $\tau \in [0,1]$. Now assume that $\bar{\beta}(\tau\|t_1\|)$, $\tilde{\beta}(\tau\|t_0\|)$ are defined and in $\overline{K}$, and $\tau \in [\rho_{j-1}, \rho_j]$. Then

$$|\overline{\beta}(\tau\|t_1\|) - \widetilde{\beta}(\tau\|t_0\|)|$$

$$= |\gamma^{(i_j)}_{\overline{\beta}(\rho_j\|t_1\|)} [\eta_\chi(\tau\|t_1\|)] - \gamma^{(i_j)}_{\overline{\beta}(\rho_j\|t_1\|)} [\eta_\theta(\tau\|t_0\|)]|$$

$$= |\gamma^{(i_j)}_{\overline{\beta}(\tau\|t_1\|)} [\eta_\chi(\tau\|t_1\|)] - \gamma^{(i_j)}_{\overline{\beta}(\tau\|t_1\|)} [\eta_\theta(\tau\|t_0\|)]|$$

$$\leq A\ |\eta_\chi(\tau\|t_1\|) - \eta_\theta(\tau\|t_0\|)| \qquad \text{[by (13-6) and (13-10)]}$$

$$\leq AC\tau\ |t_1 - t_0| \qquad \text{[by (13-8)]}$$

$$\leq AC\ |x_1 - x_0|\ .$$

(The second equality above follows because $z = \gamma_x^{(i)}(t)$ implies that $\gamma_z^{(i)} \equiv \gamma_x^{(i)}$.)

Combining this with (15-9) and setting $M = B\ \exp(LT) + AC$, we get

(15-11) (a) $$|\overline{\beta}(\tau\|t_1\|) - \beta(\tau\|t_0\|)| \leq M\ |x_1 - x_0|$$

(b) $$|\widetilde{\beta}(\tau\|t_0\|) - \beta(\tau\|t_0\|)| \leq M\ |x_1 - x_0|$$

whenever $\tau \geq \rho$, and $\bar{\beta}(\tau'\|t_1\|)$, $\tilde{\beta}(\tau'\|t_0\|)$ are defined and in $\bar{K}$ for all $\tau' \in [\tau, 1]$. (By hypothesis, $\beta(\tau\|t_0\|) \in K$ for $\tau \in [\rho, 1]$.)

It is clear that if $\bar{\beta}(\tau\|t_1\|)$, $\tilde{\beta}(\tau\|t_0\|)$ are defined and in the interior of $\bar{K}$, and $\tau > \rho$, then we can find a $\delta > 0$ such that $\bar{\beta}(\tau'\|t_1\|)$, $\tilde{\beta}(\tau'\|t_0\|)$ are defined and in $\bar{K}$ for all $\tau' \in [\tau-\delta, \tau]$. Then choosing $\varepsilon < \delta_4/M$, it is easy to show from (15-11) that $\bar{\beta}(\tau\|t_1\|)$, $\tilde{\beta}(\tau'\|t_0\|$ are defined and in $\bar{K}$ for all $\tau \in [\rho, 1]$. This completes the proof. ∎

We now prove Proposition 8.5. To prove part (1), we must show that for any compact set $K \subset \mathcal{O}$ with $K \cap C \subset V$ and any $x \in \mathcal{J}(K)$, there is an $\varepsilon > 0$ such that $|x_1 - x| < \varepsilon$ implies $x_1 \in \mathcal{J}(V)$.

Since K is compact and V is open, we can choose a compact set $\bar{K} \subset \mathcal{O}$ such that K is contained in the interior of $\bar{K}$ and $\bar{K} \cap C \subset V$. Let ε be as in Lemma 15.6. We will show that $x \in \mathcal{J}(K)$ and $|x_1 - x| < \varepsilon$ imply $x_1 \in \mathcal{J}(\bar{K})$. Since $\mathcal{J}(\bar{K}) \subset \mathcal{J}(V)$, this will prove part (1).

Assume $x \in \mathcal{J}(K)$ and $|x_1 - x| < \varepsilon$. Write $x = (t, y)$ and $x_1 = (t_1, y_1)$. Let β be any broken dependence path to x_1. By Lemma 13.4, we need only show that β can be extended to 0 and $\beta([0, \|t_1\|]) \subset \bar{K}$.

Let $\bar{\beta}$ be the broken dependence path to x_1 parallel to β. Since $x \in \mathcal{J}(K)$, Lemma 15.4 implies that $\beta(\tau\|t\|) \in K$, whenever it is defined. But it is clear from Definition 15.5 that this implies that $\overline{\beta}(\tau\|t\|)$ is defined whenever $\beta(\tau\|t_1\|)$ is. Again by Lemma 15.4, we can extend $\overline{\beta}$ to be defined on $[0, \|t\|]$ with $\overline{\beta}([0, \|t\|]) \subset K$. Applying Lemma 15.6 (with β and $\overline{\beta}$ reversed) shows that β can be extended to 0, and $\beta([0, \|t_1\|]) \subset \overline{K}$. This completes the proof of part (1).

Part (2) of the proposition is trivial, since the mapping $(\rho, (t, y)) \to \beta^{(i)}_{(t, y)}(\rho\|t\|)$ defines a continuous deformation of $\mathcal{J}(V)$ onto V, for any i.

In the proof of part (1), we also proved the following result which we will use in the next section.

Lemma 15.7: Let K, $\overline{K} \subset \mathcal{O}$ be compact sets such that K is contained in the interior of $\overline{K}$. Then the closure of $\mathcal{J}(K)$ is contained in $\mathcal{J}(\overline{K})$.

In fact, using the relation (15-11) (a) from the proof of Lemma 15.6, one can show that $\mathcal{J}(K)$ is closed. However, we will not need this result.

16. Proof of Theorem IV

The proof of Theorem IV is similar to that of Theorem I, only simpler. We first show that it suffices to prove the theorem with the $w^j(y) \equiv 0$. The method is essentially the same one used in Section 9. Define the functions $W^j(x)$ by $W^j(x) \equiv w^j[\gamma_x^{(1)}(0)]$. Since $\gamma_x^{(1)}(0) = \beta_x^{(1)}(0)$, the definition of $\mathcal{J}(V)$ shows that $\gamma_x^{(1)}(0) \in V$ for any $x \in V$. Thus, W^j is defined on $\mathcal{J}(V)$. By the analyticity of the mapping $(t, x) \to \gamma_x(t)$, $W^j \in \mathcal{A}(\mathcal{J}(V))$. The rest of the argument is the same as in Section 9.

We now write

$$a(x; D) \equiv (D_0 - \alpha^{(1)}(x) D_1) \ldots (D_0 - \alpha^{(m)}(x) D_1) + b(x; D),$$

where $b(x; D)$ is an operator on $\mathcal{O}$ of order $\leq m-1$. As in the proof of Theorem I, we will inductively define the functions u_k to be the solutions of the following sequence of Cauchy problems:

(16-1)

$$(D_0 - \alpha^{(1)}(x) D_1) \ldots (D_0 - \alpha^{(m)}(x) D_1) u_k(x) \equiv \begin{cases} v(x) & \text{if } k = 1 \\ -b(x; D) u_{k-1}(x) & \text{if } k > 1. \end{cases}$$

$$(D_0)^j u_k(0, y) \equiv 0 \quad \text{for} \quad j = 0, \ldots, m-1.$$

Then $u(x) \equiv \sum_{k=1}^{\infty} u_k(x)$ is a formal solution if (4-1) (with $w^j \equiv 0$). If the sum is absolutely convergent, then $u(x)$ is a solution function for (4-1).

We will use Lemma 12.4 to obtain the solutions u_k of (16-1). In order to be able to apply Lemma 12.4, we first introduce some notation. Let U be an open subset of $\mathbb{C}^2$ with compact closure contained in $\mathcal{O}$, such that the closure of $U \cap \mathbb{C}$ is contained in V. Let N^1 be a star-shaped neighborhood of 0 in $\mathbb{C}$ with compact closure contained in N, such that $U \subset \eta(N^1) \times \mathbb{C}$. ($N$ is as in Definition 8.2.) Since U has compact closure and η^{-1} is continuous, such an N^1 exists. Finally, let $\mathcal{J}(V; U)$ be the set $\mathcal{J}(V \cap U)$ defined by Definition 8.4 with U substituted for $\mathcal{O}$. Proposition 8.5 states that $\mathcal{J}(V; U)$ is an open subset of U.

The definitions of $\mathcal{J}(V)$ and $\mathcal{J}(V; U)$ are unchanged if η is replaced by the mapping $t \to \eta(\rho t)$ for any fixed real number $\rho > 0$. We can therefore assume that η satisfies $\left|\frac{d\eta_\theta(\tau)}{d\tau}\right| \leq 1$ whenever $\tau e^{i\theta} \in N^1$, and hence whenever $(\eta_\theta(\tau), y) \in U$ for some y.

Now let $\mathcal{L}$ be the operator $(D_0 - \alpha^{(i)}(x) D_1)$. The mapping Γ defined in Section 12 is then just the mapping $(t, y) \to \gamma_y^{(i)}(t)$. For any $x = (t, y) \in \mathcal{J}(V; U)$, we have

$$x = \gamma_x^{(i)}(t) = \gamma^{(i)}_{\beta_x^{(i)}(0)}(t)$$

$$= \Gamma(t, \beta_x^{(i)}(0)).$$

By definition of $\mathcal{J}(V;U)$, $\beta_x^{(i)}(0)$ exists and is an element of $\mathcal{J}(V;U) \cap \mathbb{C}$. Thus, condition (1) of Section 12 (page 52) is satisfied, when $\mathcal{J}(V;U)$ is substituted for U in it. Similarly, $\Gamma(\eta_\theta(\rho\tau), y)$ is just $\beta_{\Gamma(\eta_\theta(\tau), y)}^{(i)}(\rho\tau)$, so condition (2) also follows from the definition fo $\mathcal{J}(V;U)$ - again with $\mathcal{J}(V;U)$ substituted for U.

Conditions (3) and (4) follow from the restrictions on U and η made above, since $\mathcal{J}(V;U) \subset U$. We can therefore apply Lemma 12.4 to the operators $(D_0 - \alpha^{(i)}(x) D_1)$ and the open set $\mathcal{J}(V;U)$. We do so in the proof of the following result.

Lemma 16.1: Let U be an open set in $\mathbb{C}^2$ with compact closure contained in $\mathcal{O}$, such that the closure of $U \cap C$ is contained in V; let $\mathcal{J}(V;U)$ be as defined above, and let $v \in \mathcal{A}(\mathcal{J}(V))$. Then there exist solutions $u_k \in \mathcal{A}(\mathcal{J}(V;U))$ of the Cauchy problems (16-1), and constants B, L, γ and ρ such that each u_k satisfies

(16-2) $$|D_0^q D_1^r u_k(x)| \leq B \frac{L^k}{k!} E(q,r)(x)$$

for all $x \in \mathcal{J}(V;U)$ and all $q, r \in \mathbb{N}$.

Proof: Consider the Cauchy problem

(16-3) $$(D_0-\alpha^{(1)}(x)D_1) \ldots (D_0-\alpha^{(m)}(x)D_1)u(x) \equiv f(x)$$

$$(D_0)^j u(0,y) \equiv 0 \quad \text{for} \quad j = 0,\ldots,m-1.$$

As we saw in the proof of Lemma 13.2, if $U_0 \equiv f$ and U_i is defined iteratively to be the solution to the Cauchy problems

(16-4) $$(D_0-\alpha^{(i)}(x)D_1)U_i(x) \equiv U_{i-1}(x)$$

$$D_0 U_i(0,y) \equiv 0 ,$$

then $u(x) \equiv U_m(x)$ is the solution to (16-3).

Applying Lemma 12.4 to the Cauchy problems (16-4), we find that for sufficiently large constants γ and $1/\rho$: if $f \in \mathcal{A}(\mathcal{J}(V;U))$ and satisfies

$$|D_0^q D_1^r f(x)| \leq \frac{B}{k!} E(q+j+m, r)(x)$$

for all $x \in \mathcal{J}(V;U)$ and $q, r \in \mathbb{N}$; then the Cauchy problem (16-3) has a solution $u(x) \in \mathcal{A}(\mathcal{J}(V;U))$ satisfying

$$|D_0^q D_1^r u(x)| \le \frac{B}{k!} (2E_T)^m E(q+j, r)(x)$$

for all $x \in \mathcal{J}(V;U)$ and $q, r \in \mathbb{N}$.

Next, applying Lemma 12.3 to the operator $-b(x; D)$, we obtain a constant A such that

$$|D_0^q D_1^r [-b(x; D)u(x)]| \le \frac{B}{k!} A(2E_T)^m E(q+(j-1)+m, r)(x)$$

for all $x \in \mathcal{J}(V;U)$ and $q, r \in \mathbb{N}$.

We now get the solution functions $u_1, \ldots, u_k$ by repeating this procedure k times - starting with $f(x) \equiv v(x)$, then letting $f(x) \equiv -b(x; D)u_j(x)$ for $j = 1, \ldots, k-1$. The above inequalities show that <u>if</u> $v(x)$ satisfies

(16-5) $$|D_0^q D_1^r v(x)| \le \frac{B}{k!} E(q+k+m, r)(x)$$

for all $x \in \mathcal{J}(V;U)$ and $q, r \in \mathbb{N}$, <u>then</u> $u_k(x) \in \mathcal{J}(V;U)$ and satisfies (16-2) with $L = A(2E_T)^m$.

To prove (16-5), we first show that the closure of $\mathcal{J}(V;K)$ is contained in $\mathcal{J}(V)$. Let K denote this closure. Since $\mathcal{J}(V;U) \subset U$, we have $K \subset \mathcal{O}$ and $K \cap \mathbb{C} \subset V$. Choose a compact subset $\overline{K}$ of $\mathcal{O}$ such that K is contained in the interior of $\overline{K}$ and $\overline{K} \cap C \subset V$. Then Lemma 15.7 shows that the closure of $\mathcal{J}(K)$ is contained in

$\mathcal{J}(\bar{K})$, which in turn is contained in $\mathcal{J}(V)$. It is easy to see from Definition 8.4 that $\mathcal{J}(V;U) \subset \mathcal{J}(K)$. Thus, the closure of $\mathcal{J}(V;U)$ is contained in $\mathcal{J}(V)$.

Since $v(x) \in \mathcal{a}(\mathcal{J}(V))$, we can now apply Lemma 12.4 to find constants B and ρ such that

$$|D_0^q D_1^r v(x)| \leq B \frac{(q+r)!}{\rho^{q+r}}$$

for all $x \in \mathcal{J}(V;U)$ and $q, r \in \mathbb{N}$. But

$$\frac{(q+r)!}{\rho^{q+r}} \leq E(q, r)$$

$$\leq \frac{1}{k!} E(q+k, r)$$

$$\leq \frac{1}{k!} E(q+k+m, r)$$

if $\gamma \geq 1$. Thus $v(x)$ satisfies (16-5) for any k, completing the proof. ∎

The proof of Theorem IV is now straightforward. Lemma 16.1 implies that

$$|u_k(x)| \leq B \frac{L^k}{k!} E(0, 0)(x),$$

so

$$\sum_{k=1}^{\infty} |u_k(x)| \leq B \exp(L)E(0,0)(x)$$

for all $x \in \mathcal{J}(V;U)$. Therefore, the function $u(x) \equiv \Sigma\, u_k(x)$ is a solution of the Cauchy problem (4-1) (for $w^j \equiv 0$), and $u \in \mathcal{Q}(\mathcal{J}(V;U))$.

To prove that $u \in \mathcal{Q}(\mathcal{J}(V))$, we need only show that for any $x \in \mathcal{J}(V)$, there is an open subset U of $\mathcal{O}$ satisfying the hypotheses of Lemma 16.1 with $x \in \mathcal{J}(V;U)$.

Let $x \in \mathcal{J}(V)$. Then $x \in \mathcal{J}(K)$ for some compact subset K of $\mathcal{O}$ with $K \cap \mathbb{C} \subset V$. Choose a compact subset $\bar{K}$ of $\mathcal{O}$ such that $\bar{K} \cap \mathbb{C} \subset V$, and K is contained in the interior of $\bar{K}$. Let U be the interior of $\bar{K}$. Then U satisfies the hypotheses of Lemma 16.1. Moreover, it is clear from the definition of $\mathcal{J}(V;U)$ that $\mathcal{J}(K) \subset \mathcal{J}(V;U)$, so $x \in \mathcal{J}(V;U)$. This proves that $u \in \mathcal{Q}(\mathcal{J}(V))$.

The uniqueness of the solution $u(x)$ follows from the Cauchy-Kowalewski Theorem, completing the proof of Theorem IV.

IV - PROOF OF THEOREMS II AND III

To prove Theorems II and III, we will construct a Cauchy problem

$$\bar{a}(x; D)\bar{u}(x) \equiv \bar{v}(x)$$

$$(D_0)^j \bar{u}(0, y) \equiv \bar{w}^j(y)$$

which is solved by $\bar{u}(x) \equiv u \circ \sigma(x)$ if and only if $u(x)$ is a solution of (4-1). Theorem II will be proved by applying Theorem I to this Cauchy problem, and Theorem III will be proved by applying the Cauchy-Kowalewski Theorem.

The construction of this Cauchy problem, and the proof that it has the required properties, is done in Section 19, using results proved in Section 17. Section 18 is a digression to prove the existence of the characteristic functions used in Theorem I.

17. Notations and Preliminary Lemmas

We begin with some notations to be used in this chapter. For clarity, we consider "another copy" of $\mathbb{C}^{n+1}$, denoted by $\overline{\mathbb{C}}^{n+1}$. We let $\overline{x} = (\overline{x}^0, \ldots, \overline{x}^n)$ denote an element of $\overline{\mathbb{C}}^{n+1}$. Similarly, $\overline{D}_i = \partial/\partial\overline{x}^i$, etc.

We introduce the summation convention whereby any expression involving the same index as both a subscript and a superscript is to be summed on that index. The range of summation is 0 to n for a Roman letter index, and 1 to n for a Greek letter index.

Let $L(\mathbb{C}^{n+1}, \overline{\mathbb{C}}^{n+1})$ denote the set of all linear transformations from $\mathbb{C}^{n+1}$ to $\overline{\mathbb{C}}^{n+1}$. If $\tau \in L(\mathbb{C}^{n+1}, \overline{\mathbb{C}}^{n+1})$, then the $\tau^i_j \in \mathbb{C}$, for $i, j = 0, \ldots, n$, are defined by

$$\tau(p_0, \ldots, p_n) = (\tau^i_0 p_i, \ldots, \tau^i_n p_i).$$

(Remember that $\tau^i_j p_i = \sum_{i=0}^{n} \tau^i_j p_i$ by our summation convention.)

If $\sigma : U \to \mathbb{C}^{n+1}$ for some set U, then the functions $\sigma^i : U \to \mathbb{C}$ are defined by $\sigma(x) \equiv (\sigma^0(x), \ldots, \sigma^n(x))$. Similarly, if $\tau : U \to L(\mathbb{C}^{n+1}, \overline{\mathbb{C}}^{n+1})$, then $\tau^i_j : U \to \mathbb{C}$ is defined by $\tau^i_j(x) \equiv [\tau(x)]^i_j$.

In this section, we assume that $\overline{U}$ is an open neighborhood of 0 in $\overline{\mathbb{C}}^{n+1}$, and $\sigma : \overline{U} \to \mathbb{C}^{n+1}$ is any analytic mapping. Later, we will apply our results to the particular σ defined in Section 7.

Define the mapping $\sigma_* : \overline{U} \to L(\mathbb{C}^{n+1}, \overline{\mathbb{C}}^{n+1})$ by $(\sigma_*)^i_j \equiv \overline{D}_j \sigma^i$. We define the mapping σ_*^{-1} by $\sigma_*^{-1}(x) \equiv [\sigma_*(x)]^{-1}$. Then

$$\sigma_*^{-1} : \{\overline{x} \in \overline{U} : \det[\sigma_*(\overline{x})] \neq 0\} \to L(\overline{\mathbb{C}}^{n+1}, \mathbb{C}^{n+1}),$$

where $\det(\tau)$ denotes the determinant of the matrix (τ^i_j).

Given an operator $b(x; D)$ on a subset U of $\mathbb{C}^{n+1}$, we want to define an operator $\sigma^*b(\overline{x}; \overline{D})$ on a subset of $\overline{\mathbb{C}}^{n+1}$ with the property that for any analytic function f on U,

$$b(x; D)[f]\circ\sigma \equiv \sigma^*b(\overline{x}; \overline{D})[f\circ\sigma].$$

The following will turn out to be the appropriate definition.

<u>Definition 17.1:</u> Let $b(x; p)$ be an analytic function on an open subset of $T_*(\overline{\mathbb{C}}^{n+1})$. Then $\sigma_*b(\overline{x}; \overline{p})$ is the function on a subset of $T_*(\overline{\mathbb{C}}^{n+1})$ defined by

(17-1) $$\sigma^*b(\overline{x}; \overline{p}) \equiv b(\sigma(\overline{x}); [\sigma_*^{-1}(\overline{x})](\overline{p})).$$

It follows immediately from this definition that:

(1) $\sigma^*b(\overline{x}; \overline{p})$ is analytic on the open subset of $T_*(\overline{\mathbb{C}}^{n+1})$ consisting of all points $(\overline{x}, \overline{p})$ satisfying

(i) $\overline{x} \in \overline{U}$, $\det[\sigma_*(\overline{x})] \neq 0$

(ii) $(\sigma(\bar{x}),\ [\sigma_*^{-1}(\bar{x})](\bar{p}))$ is in the domain of $b(x;p)$

(2) If $b(x;p)$ is homogeneous in p, then $\sigma^*b(\bar{x};\bar{p})$ is homogeneous in $\bar{p}$ of the same degree.

(3) If $b(x;p)$ is a polynomial in p, then $\sigma^*b(\bar{x};\bar{p})$ is a polynomial in $\bar{p}$ of the same degree.

For an operator $b(x;D)$, $\sigma^*b(\bar{x};\bar{D})$ is thus formed by substituting $\sigma(\bar{x})$ for x and $(\sigma_*^{-1})_i^j(\bar{x})\bar{D}_j$ for p_i in $b(x;p)$. Lemma 17.3 will show that $\sigma^*b(\bar{x};\bar{D})$ is a well-defined operator (i.e., it is independent of the order of multiplication of the p_i in $b(x;p)$), and that is satisfies (17-1). The proof requires the following simple result.

<u>Lemma 17.2</u>: Let $\bar{x}_0 \in \bar{U}$, with $\det[\sigma_*(\bar{x}_0)] \neq 0$, and let f be a function analytic at $\sigma(\bar{x}_0) \in \mathbb{C}^{n+1}$. Then

$$(Df)\circ\sigma(\bar{x}_0) = (\sigma_*^{-1})(\bar{x}_0)[\bar{D}(f\circ\sigma)(\bar{x}_0)]$$

<u>Proof</u>: By the chain rule,

$$\bar{D}_i(f\circ\sigma)(\bar{x}_0) = [\bar{D}_i\sigma^j](\bar{x}_0)\cdot[D_jf](\sigma(\bar{x}_0)).$$

By definition of σ_*, this becomes

$$\bar{D}(f\circ\sigma)(\bar{x}_0) = \sigma_*(\bar{x}_0)[(Df)\circ\sigma(\bar{x}_0)]\ .$$

Applying $\sigma_*^{-1}(\bar{x}_0)$ to this equality proves the lemma. ∎

Lemma 17.3: Let $b(x; D)$ be an operator on an open set N in $\mathbb{C}^{n+1}$. Then

(1) $\sigma^* b(\bar{x}; \bar{D})$ is a well-defined operator on

$$\bar{N} = \{\bar{x} \in \bar{U} \cap \sigma^{-1}(N) : \det[\sigma_*(\bar{x})] \neq 0\}$$

(2) If $\bar{x} \in \bar{U}$ and the function f is analytic at $\sigma(\bar{x})$, then

$$[b(x; D)f][\sigma(\bar{x})] = \sigma^* b(\bar{x}; \bar{D})[f \circ \sigma](\bar{x}).$$

(3) If $k(x; p)$ is the principal part of $b(x; D)$, then $\sigma^* k(\bar{x}; \bar{p})$ is the principal part of $\sigma^* b(x; D)$.

Proof: (2) Write $b(x; D) = \Sigma\, b_{i_1 \ldots i_n} D_{i_1} \ldots D_{i_k}$. Then

$$[b(x; D)f] \circ \sigma(\bar{x}) = \Sigma\, b_{i_1 \ldots i_k}[\sigma(\bar{x})][D_{i_1 \ldots i_k} f] \circ \sigma(\bar{x})$$

and

$$\sigma^* b(\bar{x};\bar{D})[f\circ\sigma](\bar{x}) =$$

$$\Sigma\, b_{i_1 \ldots i_k}[\sigma(\bar{x})]([\sigma_*^{-1}(\bar{x})]_{i_1}^{j_1}\bar{D}_{j_1}) \ldots ([\sigma_*^{-1}(\bar{x})]_{i_k}^{j_k}\bar{D}_{j_k})[f\circ\sigma](\bar{x}).$$

Applying Lemma 17.2 k times shows that these two expressions are equal.

(1) Let $\bar{x} \in \bar{N}$, and g a function analytic at $\bar{x}$. Since $\det[\sigma_*(\bar{x})] \neq 0$, we can find a unique analytic inverse σ^{-1} on a neighborhood of $\sigma(\bar{x})$. Then (2) implies

$$\sigma^* b(\bar{x};\bar{D})[g](\bar{x}) = [b(x;D)(g\circ\sigma^{-1})]\circ\sigma(\bar{x}),$$

so $\sigma^* b(\bar{x};\bar{D})$ is well-defined on $\bar{N}$.

(3) This is an immediate consequence of Definition 17.1. ∎

The application of this result to the proof of Theorems II and III will be by means of the following lemma.

<u>Lemma 17.4:</u> Assume $\sigma(\bar{U}\cap\bar{\mathbb{C}}^n) \subset \mathbb{C}^n$, and that $\det[\sigma_*(0,y)]$ is not identically zero on a neighborhood of 0 in $\bar{\mathbb{C}}^n$. Let $b(x;D)$, $b^i(x;D)$ be operators on a neighborhood of 0 in $\mathbb{C}^{n+1}$. Then a function u is a solution of the Cauchy problem

$$b(x; D)u(x) \equiv v(x)$$

$$b^i(x; D)u(0, y) \equiv w^i(y), \qquad i = 0, \ldots, m-1$$

if and only if $\bar{u} \equiv u \circ \sigma$ is a solution of the Cauchy problem

$$\sigma^* b(\bar{x}; \bar{D})\bar{u}(\bar{x}) \equiv v \circ \sigma(\bar{x})$$

$$\sigma^* b^i(\bar{x}; \bar{D})\bar{u}(0, \bar{y}) \equiv w^i \circ \sigma(0, y), \qquad i = 0, \ldots, m-1.$$

Proof: The "only if" part follows immediately from part (2) of Lemma 17.3. For the converse, the same lemma implies that if $\bar{u}$ is a solution of the second Cauchy problem, then u is a solution of the first in the neighborhood of any point $\sigma(0, \bar{y})$ in its domain with $\det[\sigma^*(0, y)] \neq 0$. By hypothesis, such a point exists. The result then follows by analytic continuation. ∎

We now examine the relationship between the solutions to the bicharacteristic equations for $b(x; D)$ and $\sigma^* b(\bar{x}; \bar{D})$. In particular, we will show that if $t \to \bar{\xi}(t)$ is a bicharacteristic curve of $\sigma^* b$, then $t \to \sigma \circ \bar{\xi}(t)$ is a bicharacteristic curve of b.

Before doing this, we prove the following result which we will need.

Lemma 17.5: If $\bar{x} \in \bar{U}$ and $\det[\sigma_*(\bar{x})] \neq 0$, then

$$\bar{D}_r(\sigma_*^{-1})_i^{\ell}(\bar{x}) = (\sigma_*^{-1})_i^j(\bar{x}) \cdot (\sigma_*)_r^k(\bar{x}) \cdot \bar{D}_j(\sigma_*^{-1})_k^{\ell}(\bar{x}).$$

Proof: For convenience, we drop the reference to the point $\bar{x}$. Since $(\sigma_*)_j^k = \bar{D}_j \sigma^k$, we see that

$$\text{(17-2)} \qquad \bar{D}_r(\sigma_*)_j^k \cdot (\sigma_*^{-1})_k^{\ell} = \bar{D}_j(\sigma_*)_r^k \cdot (\sigma_*^{-1})_k^{\ell}.$$

But

$$\bar{D}_s[(\sigma_*)_q^k \cdot (\sigma_*^{-1})_k^{\ell}] = \bar{D}_s[\delta_q^{\ell}] = 0$$

implies

$$\bar{D}_s(\sigma_*)_q^k(\sigma_*^{-1})_k^{\ell} = -(\sigma_*)_q^k \bar{D}_s(\sigma_*^{-1})_k^{\ell} .$$

Applying this to both sides of (17-2) yields

$$(\sigma_*)_j^k \cdot \bar{D}_r(\sigma_*^{-1})_k^{\ell} = (\sigma_*)_r^k \cdot \bar{D}_j(\sigma_*^{-1})_k^{\ell} .$$

Multiplying this last equality by $(\sigma_*^{-1})_i^j$ (and summing on j) gives the required result. ∎

Lemma 17.6: Let $g(x; p)$ be an analytic function on a subset of $T_*(\mathbb{C}^{n+1})$ which is homogeneous in p, and let $\bar{x}_0 \in \bar{U}$ with $\det[\sigma_*(\bar{x}_0)] \neq 0$.

If $t \to (\bar{\xi}(t); \bar{\pi}(t))$ is the solution of the bicharacteristic equations of $\sigma^*g(\bar{x}; \bar{p})$ with the initial conditions $\bar{\xi}(0) = \bar{x}_0$, $\bar{\pi}(0) = \bar{p}_0$,

then

$$t \to (\xi(t); \pi(t)) = (\sigma \circ \bar{\xi}(t); [\sigma_*^{-1}(\bar{\xi}(t))][\bar{\pi}(t)])$$

is the solution of the bicharacteristic equations of $g(x; p)$ with initial conditions

$$\xi(0) = \sigma(\bar{x}_0), \qquad \pi(0) = \sigma_*^{-1}(x_0)(\bar{p}_0) .$$

Moreover, $g(\xi(t); \pi(t)) \equiv \sigma^*g(\bar{\xi}(t); \bar{\pi}(t))$. (Thus, $t \to (\xi(t), \pi(t))$ is a bicharacteristic strip if $t \to (\bar{\xi}(t), \bar{\pi}(t))$ is.)

Proof: We must show that for each $i = 0, \ldots, n$:

(17-3) $$\frac{d\xi^i}{dt}(t) = \frac{\partial g}{\partial p_i}[\xi(t); \pi(t)]$$

$$\frac{d\pi_i}{dt}(t) = -(D_i g)[\xi(t), \pi(t)] \tag{17-4}$$

whenever $\det[\sigma(\bar{\xi}(t))] \neq 0$. We now compute:

$$\frac{d\xi^i}{dt}(t) = \frac{d[\sigma^i \circ \bar{\xi}]}{dt}(t)$$

$$= D_j\sigma^i[\bar{\xi}(t)] \cdot \frac{d\bar{\xi}^j}{dt}(t)$$ [by the chain rule]

$$= D_j\sigma^i[\bar{\xi}(t)] \cdot \frac{\partial[\sigma^* g]}{\partial p_j}(\bar{\xi}(t), \bar{\pi}(t))$$

[since $\bar{\xi}$, $\bar{\pi}$ satisfy bicharacteristic equations]

$$= D_j\sigma^i[\bar{\xi}(t)] \cdot \frac{\partial g}{\partial p_\ell}[\sigma \circ \bar{\xi}(t); [\sigma_*^{-1}(\bar{\xi}(t))](\bar{\pi}(t))] \cdot (\sigma_*^{-1})^j_\ell(\bar{\xi}(t))$$

[by definition of $\sigma^* g$, and the chain rule]

$$= (\sigma_*)^i_j[\bar{\xi}(t)] \cdot (\sigma_*^{-1})^j_\ell[\bar{\xi}(t)] \cdot \frac{\partial g}{\partial p_\ell}[\xi(t); \pi(t)]$$

[by definition of ξ, π and σ_*]

$$= \frac{\partial g}{\partial p_i}(\xi(t), \pi(t)),$$ [by definition of σ^{-1}],

thus proving (17-3).

To verify (17-4), we compute

$$\frac{d\pi_i}{dt}(t) = \frac{d}{dt}[(\sigma_*^{-1})_i^j[\bar{\xi}(t)] \cdot \bar{\pi}_j(t)]$$

$$= (\sigma_*^{-1})_i^j[\bar{\xi}(t)] \cdot \frac{d\pi_j}{dt}(t)$$

$$+ \bar{D}_\ell[(\sigma_*^{-1})_i^j][\bar{\xi}(t)] \cdot \frac{d\bar{\xi}^\ell}{dt}(t) \cdot \bar{\pi}_j(t)$$

$$= (\sigma_*^{-1})_i^j[\bar{\xi}(t)] \cdot [- \bar{D}_j(\sigma^* g)(\bar{\xi}(t), \bar{\pi}(t))]$$

$$+ \bar{D}_\ell[(\sigma_*^{-1})_i^j][\bar{\xi}(t)] \cdot \frac{d\bar{\xi}^\ell}{dt}(t) \cdot \bar{\pi}_j(t)$$

[since $\bar{\xi}, \bar{\pi}$ satisfy bicharacteristic equations]

$$= (\sigma_*^{-1})_i^j[\bar{\xi}(t)] \cdot \{- \bar{D}_j\sigma^\ell[\bar{\xi}(t)] \cdot D_\ell g(\sigma \circ \bar{\xi}(t); \sigma_*^{-1}(\bar{\xi}(t))[\bar{\pi}(t)])$$

$$- \frac{\partial g}{\partial p_k}(\sigma \circ \bar{\xi}(t); \sigma_*^{-1}(\bar{\xi}(t))[\bar{\pi}(t)] \cdot \bar{D}_j[(\sigma_*^{-1})_k^\ell][\bar{\xi}(t)] \cdot \bar{\pi}_\ell(t)\}$$

$$+ \bar{D}_\ell[(\sigma_*^{-1})_i^j][\bar{\xi}(t)] \cdot \frac{d\bar{\xi}^\ell}{dt}(t) \cdot \bar{\pi}_j(t)$$

[by definition of $\sigma^* g$, and the chain rule]

$$= - (\sigma_*^{-1})_i^j[\bar{\xi}(t)] \cdot \bar{D}_j\sigma^\ell[\bar{\xi}(t)] \cdot D_\ell g(\xi(t); \pi(t))$$

$$- (\sigma_*^{-1})_i^j[\bar{\xi}(t)] \cdot \frac{d\xi^k}{dt}(t) \cdot \bar{D}_j[(\sigma_*^{-1})_k^\ell][\bar{\xi}(t)] \cdot \bar{\pi}_\ell(t)$$

$$+ \bar{D}_\ell[(\sigma_*^{-1})_i^j][\bar{\xi}(t)] \cdot \frac{d\bar{\xi}^\ell}{dt}(t) \cdot \bar{\pi}_j(t)$$

[by definition of ξ, π, and (17-3)]

$$= - D_i g(\xi(t); \pi(t))$$

$$- (\sigma_*^{-1})_i^j[\bar{\xi}(t)] \cdot \bar{D}_r \sigma^k[\bar{\xi}(t)] \cdot \frac{d\bar{\xi}^r}{dt}(t) \cdot \bar{D}_j[(\sigma_*^{-1})_k^\ell][\bar{\xi}(t)] \cdot \bar{\pi}_\ell(t)$$

$$+ \bar{D}_\ell[(\sigma_*^{-1})_i^j][\bar{\xi}(t)] \cdot \frac{d\bar{\xi}^\ell}{dt}(t) \cdot \bar{\pi}_j(t)$$

[by definition of σ_*^{-1} and ξ, plus the chain rule]

$$= - D_i g(\xi(t), \pi(t))$$

$$+ [\bar{D}_r(\sigma_*^{-1})_i^\ell - (\sigma_*^{-1})_i^j \cdot \bar{D}_r \sigma^k \cdot D_j(\sigma_*^{-1})_k^\ell][\bar{\xi}(t)] \cdot$$

$$\frac{d\bar{\xi}^r}{dt}(t) \cdot \bar{\pi}_\ell(t)$$

$$= - D_i g(\xi(t); \pi(t)) \qquad \text{[by Lemma 17.5].}$$

This proves (17-4).

The fact that

$$g(\xi(t); \pi(t)) \equiv \sigma^* g(\bar{\xi}(t); \bar{\pi}(t))$$

is an immediate consequence of the definitions of $\xi(t)$, $\pi(t)$ and $\sigma^* g$. $\blacksquare$

18. Existence of the Characteristic Functions

We now construct the characteristic functions used in Theorem I. The basic idea is to first find mappings σ, π such that $t \to (\sigma(t, y); \pi(t, y))$ is a bicharacteristic strip for $h(x; p)$. Hence, $h(\sigma(t, y); \pi(t, y)) \equiv 0$. We then find a function $\varphi(x)$ with $D\varphi[\sigma(t, y)] \equiv \pi(t, y)$, which will imply that φ is a characteristic function of h.

The bicharacteristic strips are easy to construct. We just solve the ordinary differential equations

$$(18\text{-}1) \qquad \frac{\partial \sigma^i}{\partial t}(t, y) \equiv \frac{\partial h}{\partial p_i}(\sigma(t, y); \pi(t, y))$$

$$\frac{\partial \pi_i}{\partial t}(t, y) \equiv - \frac{\partial h}{\partial x^i}(\sigma(t, y); \pi(t, y))$$

with initial values $\sigma(0, y)$, $\pi(0, y)$ satisfying

$$h(\sigma(0, y); \pi(0, y)) \equiv 0.$$

A simple computation using (18-1) shows that

$$\frac{\partial}{\partial t} [h(\sigma(t, y); \pi(t, y))] \equiv 0,$$

so we have

(18-2) $$h(\sigma(t, y); \pi(t, y)) \equiv 0.$$

Proposition 18. 2 will enable us to construct the characteristic function $\varphi(x)$, given σ and π. First, we prove it for the special case in which σ is the identity mapping. We assume that $h(x; p)$ is analytic and homogeneous in p on a subset of $T_*(\mathbb{C}^{n+1})$.

Lemma 18.1: Let N be an open subset of $\mathbb{C}^n$, U an open neighborhood of N in $\mathbb{C}^{n+1}$, and $\pi : U \to \mathbb{C}^{n+1}$ an analytic mapping such that $t \to (t, y; \pi(t, y))$ is a bicharacteristic strip of $h(x; p)$ for each $y \in N$. Let $\psi(y) \in \mathcal{A}(N)$ be such that $D_\mu \psi(y) \equiv \pi_\mu(0, y)$ for $\mu = 1, \ldots, n$. Then there exists an analytic function $\varphi(x)$ on a neighborhood of N in $\mathbb{C}^{n+1}$ such that $\varphi(0, y) \equiv \psi(y)$, and $D\varphi(x) \equiv \pi(x)$.

Proof: Let $\varphi(t, y)$ be the solution to the ordinary differential equation

(18-3) $$\frac{\partial}{\partial t} \varphi(t, y) = \pi_0(t, y) - h(t, y; \pi(t, y))$$

$$\varphi(0, y) = \psi(y).$$

It is clear that $\varphi(x)$ is defined and analytic on a neighborhood of N. Moreover, for $\mu = 1, \ldots, n$, we have:

$$\frac{\partial}{\partial y^\mu}\left[\frac{\partial\varphi}{\partial t}(t,y)\right] \equiv \frac{\partial}{\partial t}[D_\mu\varphi](t,y)$$

$$\equiv D_\mu \pi_0(t,y) - \frac{\partial h}{\partial x^\mu}(t,y;\pi(t,y))$$

$$- \frac{\partial h}{\partial p_j}(t,y;\pi(t,y)) \cdot D_\mu \pi_j(t,y).$$

But the bicharacteristic equations imply that

$$- \frac{\partial h}{\partial x^i}(t,y;\pi(t,y)) \equiv \frac{\partial \pi_i}{\partial t}(t,y)$$

$$\frac{\partial h}{\partial p_j}(t,y;\pi(t,y)) \equiv \delta_0^j .$$

Hence, the above equality yields

$$\frac{\partial}{\partial t}[D_\mu\varphi](t,y) \equiv \frac{\partial}{\partial t}\pi_\mu(t,y)$$

for $\mu = 1,\ldots,n$. Since $D_\mu\varphi(0,y) \equiv D_\mu\psi(0,y) \equiv \pi_\mu(0,y)$, this shows that $D_\mu\varphi \equiv \pi_\mu$ for $\mu = 1,\ldots,n$.

Now observe that because $t \to (t,y;\pi(t,y))$ is a bicharacteristic strip, $h(t,y;\pi(t,y)) \equiv 0$. Hence, (18-3) shows that $D_0\varphi(x) \equiv \pi_0(x)$, completing the proof that $D\varphi \equiv \pi$. ∎

<u>Proposition 18.2:</u> Let N be an open subset of $\mathbb{C}^n$, U an open

neighborhood of N in $\mathbb{C}^{n+1}$, and σ, $\pi : U \to \mathbb{C}^{n+1}$ analytic mappings such that for each $y \in N$:

(1) $t \to (\sigma(t, y); \pi(t, y))$ is a bicharacteristic strip of $h(x; p)$.

(2) $\sigma(0, y) = y$.

(3) $\det[\sigma_*(0, y)] \neq 0$.

Let $\psi(y) \in \mathcal{A}(N)$ with $D_\mu \psi(y) \equiv \pi_\mu(0, y)$ for $\mu = 1, \ldots, n$.

Then there exists an analytic function $\varphi(x)$ on a neighborhood of N in C^{n+1} such that $\varphi(0, y) \equiv \psi(y)$ and $D\varphi \circ \sigma(x) \equiv \pi(x)$.

<u>Proof</u>: Since $\det[\sigma_*(0, y)] \neq 0$ on N, σ is 1-1 with an analytic inverse on some neighborhood $\bar{U}$ of N in $\mathbb{C}^{n+1}$. Let $\overline{h}(\bar{x}; \bar{p})$ $\equiv \sigma^* h(x; p)$. Then $h(x; p) \equiv (\sigma^{-1})^* \overline{h}(\bar{x}; \bar{p})$.

Applying Lemma 17.6 to the mapping σ^{-1}, we see that for each $y \in N$,

$$t \to (t, y; \bar{\pi}(t, y))$$

is a bicharacteristic strip of $\overline{h}(\bar{x}; \bar{p})$, where

$$(18\text{-}3) \qquad \bar{\pi}(t, y) \equiv \sigma_*(t, y)[\pi(t, y)]$$

(since $(\sigma^{-1})_*[\sigma(x)] = \sigma_*^{-1}(x)$).

Because $\sigma(0, y) \equiv y$, we get $(\sigma_*(0, y))^j_\mu \equiv D_\mu \sigma^j(0, y) \equiv \delta^j_\mu$ for each $j = 0, \ldots, n$ and $\mu = 1, \ldots, n$. Equation (18-3) then implies that

$$\overline{\pi}_\mu(0, y) \equiv \pi_\mu(0, y)$$

for $\mu = 1, \ldots, n$.

We can now apply Lemma 18.1 to $\overline{h}(\overline{x}; \overline{p})$, $\overline{\pi}$ and ψ, since $D_\mu \psi(y) \equiv \pi(0, y) \equiv \overline{\pi}(0, y)$, to get a function $\overline{\varphi}(\overline{x})$ such that $\overline{\varphi}(0, y) \equiv \psi(y)$ and $\overline{D}\,\overline{\varphi}(\overline{x}) \equiv \overline{\pi}(\overline{x})$. Let $\varphi(x) \equiv \overline{\varphi} \circ \sigma^{-1}(x)$. Then for any $\overline{x} \in \overline{U}$,

$$D\varphi \circ \sigma(\overline{x}) = \sigma_*^{-1}(\overline{x})[\overline{D}(\varphi \circ \sigma)(\overline{x})] \qquad \text{[by Lemma 17.2]}$$

$$= \sigma_*^{-1}(\overline{x})[\overline{D}\,\overline{\varphi}(\overline{x})]$$

$$= \sigma_*^{-1}(\overline{x})[\overline{\pi}(\overline{x})]$$

$$= \pi(\overline{x}) \qquad \text{[by (18-3)]},$$

proving that $D\varphi \circ \sigma \equiv \pi$. ∎

We can now construct the characteristic functions. Recall that we assumed a neighborhood N of 0 in $\mathbb{C}^n$ and a function $\alpha(y) \in \mathcal{A}(N)$ such that

(18-4) $$h(0, y; \alpha(y), 1, 0, \ldots, 0) \equiv 0\ ,$$

and for all $y \in N$:

(18-5) $$\frac{\partial h}{\partial p_0}(0, y; \alpha(y), 1, 0, \ldots, 0) \neq 0.$$

We must construct a function $\varphi(x)$ analytic on a neighborhood of N in $\mathbb{C}^{n+1}$ such that

(18-6) (a) $$\varphi(0, y^1, \ldots, y^n) \equiv y^1$$

(b) $$D_0\varphi(0, y) \equiv \alpha(y)$$

(c) $$h(x; D\varphi(x)) \equiv 0.$$

Define the mappings σ, π by letting

$$t \to (\sigma(t, y); \pi(t, y))$$

be the solution path for the bicharacteristic equations of $h(x; p)$ with initial conditions

$$(18\text{-}7) \qquad \sigma(0, y) \equiv y$$

$$\pi(0, y) \equiv (\alpha(y), 1, 0, \ldots, 0).$$

By (18-4), this path is a bicharacteristic strip of h.

The mappings σ, π are analytic on a neighborhood of N in $\mathbb{C}^{n+1}$. Using (18-7) and the bicharacteristic equations of $h(x; p)$, we get

$$(\sigma_*)^j_\mu(0, y) \equiv D_\mu \sigma^j(0, y) \equiv \delta^j_\mu$$

$$(\sigma_*)^0_0(0, y) \equiv D_0 \sigma^0(0, y) \equiv \frac{\partial h}{\partial p_0}(0, y; \alpha(y), 1, 0, \ldots, 0).$$

Then (18-5) implies that for all $y \in N$, $\det[\sigma_*(0, y)] \neq 0$.

We now apply Proposition 18.2 with $\psi(y^1, \ldots, y^n) \equiv y^1$ to get the function $\varphi(x)$ analytic on a neighborhood of N and satisfying $\varphi(0, y) \equiv \psi(y)$, $D\varphi \circ \sigma(x) \equiv \pi(x)$. Then (18-6) (a) follows from our choice of ψ and (18-6) (b) follows from (18-7). We also have

$$h(\sigma(t, y); D\varphi[\sigma(t, y)]) \equiv h(\sigma(t, y); \pi(t, y)) \equiv 0$$

by our choice of σ and π. But $\det[\sigma_*(0, y)] \neq 0$ for $y \in N$ implies that the range of σ is an open neighborhood of N, so the above equality proves (18-6) (c). Hence, φ is the required characteristic function.

19. Proof of Theorems II and III

We now return to the proof of Theorems II and III. We assume the notation of Section 7, so $h(x; p)$ is the principal part of the $m^{\underline{th}}$ order operator $a(x; D)$, and $g(x; p) \equiv h(x; p)/(p_0)^{m-1}$ is homogeneous of degree 1. The mappings σ, π are defined so that

$$t \to (\sigma(t, y); \pi(t, y))$$

is the solution of the bicharacteristic equations of $g(x; p)$ with

$$\text{(19-1)} \qquad \sigma(0, y) \equiv y$$

$$\pi(0, y) \equiv \delta^0 = (1, 0, \ldots, 0).$$

For convenience, we will let $g(x)$ denote $g(x; \delta^0)$.

To find the function $\overline{u} \equiv u \circ \sigma$, we would like to solve the Cauchy problem

$$\sigma^* a(\overline{x}; \overline{D})\overline{u} \equiv v \circ \sigma$$

$$\sigma^*(D_0)^j \overline{u}(0, y) \equiv w^j(y) .$$

However, we will see that $\det[\sigma_*(0, y)] = 0$ if y is a characteristic point of $\mathbb{C}^n$ for $a(x; D)$. Thus, σ^*a and $\sigma^*(D_0)^j$ are not defined at a characteristic point of $\mathbb{C}^n$, so we cannot solve this Cauchy problem for $\bar{u}$. Instead, we will solve the following equivalent problem:

(19-2) $$\sigma^*[(g(x))^{m-1}a(x; D)]\bar{u} \equiv [(g(x))^{m-1}v(x)]\circ\sigma$$

$$\sigma^*[(g(x))^j(D_0)^j]\bar{u}(0, y) \equiv [g(0, y)]^j w^j(y).$$

It turns out that these operators are defined and analytic on a neighborhood of 0 in $\mathbb{C}^{n+1}$, and 0 is not a characteristic point for the above problem even if it is one for the original problem (4-1).

We first prove some simple results about the mapping σ.

Lemma 19.1: For each $i = 0, \ldots, n$ and $\mu = 1, \ldots, n$:

(1) $(\sigma_*)^i_\mu(0, y) \equiv \delta^i_\mu$

(2) $(\sigma_*)^0_0(0, y) \equiv g(0, y)$

(3) $(\sigma_*)^\mu_0(0, y) \equiv \dfrac{\partial g}{\partial p_\mu}(0, y; \delta^0)$

Proof: (1) follows from (19-1), and (3) follows from the bicharacteristic equations. For (2), we have

$$(\sigma_*)^0_0(0, y) \equiv D_0\sigma^0(0, y)$$

$$\equiv \frac{\partial g}{\partial p_0}(0, y; \delta^0)$$

by the bicharacteristic equations. But, the homogeneity of $g(x; p)$ implies

$$g(0, y; \delta^0) \equiv \delta^0_i \frac{\partial g}{\partial p_i}(0, y; \delta^0)$$

$$\equiv \frac{\partial g}{\partial p_0}(0, y; \delta^0),$$

proving (2). ∎

Lemma 19.2: For each $i = 0, \ldots, n$ and $\mu = 1, \ldots, n$:

(1) $\det[\sigma_*(0, y)] \equiv g(0, y)$

(2) (a) $(\sigma_*^{-1})^i_\mu(0, y) \equiv \delta^i_\mu$

(b) $(\sigma_*^{-1})^0_0(0, y) \equiv 1/g(0, y)$

(c) $(\sigma_*^{-1})^\mu_0(0, y) \equiv - \frac{\partial g}{\partial p_\mu}(0, y; \delta^0)/g(0, y)$

(3) For some neighborhood $\overline{V}$ of 0 in $\overline{\mathbb{C}}^{n+1}$:

(a) $(\sigma_*^{-1})^i_\mu \in \mathcal{A}(\overline{V})$

(b) $(g \circ \sigma) \cdot (\sigma_*^{-1})^i_0 \in \mathcal{A}(\overline{V})$.

Proof: (1) and (2) are easily verified using Lemma 19.1. By (1), we can find a neighborhood $\overline{V}$ of 0 and a function $f \in \mathcal{A}(\overline{V})$ such that

$$\det[\sigma_*(\overline{x})] = f(\overline{x}) g \circ \sigma(\overline{x})$$

$$f(\overline{x}) \neq 0$$

for all $\overline{x} \in \overline{V}$. Part (3) then follows from part (2), plus the fact that $\det[\sigma_*(\overline{x})](\sigma_*^{-1})^j_i(\overline{x})$ is the minor of $(\sigma_*)^i_j(\overline{x})$ in the matrix of $\sigma_*(\overline{x})$ and is thus analytic on $\overline{V}$. ∎

We now prove that the operators in the Cauchy problem (19-2) have the required properties.

Lemma 19.3: For each $\mu = 1, \ldots, n$, $\sigma^*(D_\mu)$ is an analytic operator on a neighborhood of 0 in $\overline{\mathbb{C}}^{n+1}$.

Proof: By definition of σ^*, we have

$$\sigma^*(D_\mu) \equiv (\sigma_*^{-1})^i_\mu \, \overline{D}_i \, .$$

The lemma is thus an immediate consequence of part (3-a) of Lemma 19.2. ∎

Lemma 19.4: For any integer j,

(1) $\sigma^*[(g(x))^j \cdot (D_0)^j]$ is an analytic operator on a neighborhood of 0 in $\overline{\mathbb{C}}^{n+1}$.

(2) $\sigma^*[(g(x))^j \cdot D_0)^j](0, \overline{y}) \equiv (\overline{D}_0)^j + b(y; \overline{D})$ for some operator $b(y; \overline{D})$ of order less than j in $\overline{D}_0$.

Proof: (1) By part (3-b) of Lemma 19.2, we can find a neighborhood $\overline{V}$ of 0 in $\overline{\mathbb{C}}^{n+1}$ and functions $f^i(\overline{x}) \in \mathcal{O}(\overline{V})$ such that

$$(\sigma_*^{-1})^i_0(\overline{x}) \equiv f^i(\overline{x})/g \circ \sigma(\overline{x})$$

for each $i = 0, \ldots, n$. Then

$$\sigma^*[(g(x))^j \cdot (D_0)^j] \equiv [g \circ \sigma(\bar{x})]^j [(\sigma_*^{-1})_0^i(\bar{x}) \cdot \bar{D}_i]^j$$

$$\equiv [g \circ \sigma(\bar{x})]^j [\frac{f^i(\bar{x})}{g \circ \sigma(\bar{x})} \cdot \bar{D}_i]^j ,$$

which is clearly an analytic operator on $\bar{V}$.

(2) By part (3) of Lemma 17.2, the coefficient of $(\bar{D}_0)^j$ in $\sigma^*[(g(x))^j \cdot (D_0)^j]$ is obtained by setting $\bar{p} = \delta^0$ in $\sigma^*[(g)^j \cdot (p_0)^j](\bar{x}; \bar{p})$. We thus calculate

$$\sigma^*[(g)^j \cdot (p_0)^j](0, y; \delta^0)$$

$$\equiv [g \circ \sigma(0, y)]^j \cdot [(\sigma_*^{-1})_0^i(0, y) \cdot \delta_i^0]^j$$

$$\equiv [g(0, y)]^j \cdot [1/g(0, y)]^j \qquad \text{[by Lemma 19.2 (2-6)]}$$

$$\equiv 1.$$

Thus, the coefficient of $(\bar{D}_0)^j$ in the $j^{\underline{th}}$ order operator $\sigma^*[(g(x))^j(D_0)^j]$ is 1 on $\mathbb{C}^n$. This proves (2). ∎

<u>Lemma 19.5:</u> If $b(x; D)$ is an operator on a neighborhood of 0 in

$\mathbb{C}^{n+1}$ which is of order $\leq j$ in D_0, then $\sigma^*[(g(x))^j \cdot b(x; D)]$ is an analytic operator on a neighborhood of 0 in $\overline{\mathbb{C}}^{n+1}$.

Proof: Write $b(x; D)$ as a sum of the form

$$b(x; D) \equiv \sum_k b_i^{\mu_1 \cdots \mu_k}(x) \cdot (D_0)^i D_{\mu_1} \cdots D_{\mu_k}$$

for analytic functions $b_i^{\mu_1 \cdots \mu_k}$. Then

$$\sigma^*[(g(x))^j \cdot b(x; D)] \equiv$$

$$\sum b_i^{\mu_1 \cdots \mu_k}[\sigma(\overline{x})] \cdot [g \circ \sigma(\overline{x})]^{j-i} \cdot \sigma_*[(g(x))^i (D_0)^i] \cdot$$

$$\sigma_*(D_{\mu_1}) \cdots \sigma_*(D_{\mu_k}),$$

and the result follows from the preceding two lemmas. ∎

Recall that in Section 7 we defined $\tilde{h}(x; p)$ such that

(19-3) $$\tilde{h}(x; g(x) \cdot p_0, p_1, \ldots, p_n) \equiv [g(x)]^{m-1} \cdot h(x; p).$$

Theorem II assumes that $\tilde{h}(x; p)$ has constant multiplicity in the direction of $\mathbb{C}^{n-1}$. We let $\tilde{h}_i(x; p)$ denote the homogeneous polynomials in p with

$$\tilde{h}(x;p) \equiv \tilde{h}_1(x;p) \ldots \tilde{h}_s(x;p)$$

of Definition 5.1.

The following result shows that the operator $\sigma^*[(g(x))^{m-1}a(x;D)]$ in (19-2) has the desired properties, and explains why we introduced $\tilde{h}(x;p)$.

Lemma 19.6:

(1) $\sigma^*[(g(x))^{m-1} \cdot h(x;p)](0,y) \equiv$

$$\tilde{h}(0,y;\bar{p}_0 - \frac{\partial g}{\partial p_\mu}(0,y;\delta^0)\cdot\bar{p}_\mu, \bar{p}_1, \ldots, \bar{p}_n)$$

(2) With the hypothesis of Theorem II,

$$\sigma^*[(g(x))^{m-1} \cdot h(x;p)] \equiv [\bar{h}_1(\bar{x};\bar{p})]^{k_1} \ldots [\bar{h}_s(\bar{x};\bar{p})]^{k_s},$$

where the $\bar{h}_i(x;p)$ are polynomials in $\bar{p}$ with coefficients analytic on a neighborhood of 0 in $\bar{\mathbb{C}}^{n+1}$, which satisfy

$$\bar{h}_i(0, y; \bar{p}) \equiv \tilde{h}_i(0, y; \bar{p}_0 - \frac{\partial g}{\partial p_\mu}(0, y; \delta^0)\bar{p}_\mu, \bar{p}_1, \ldots, \bar{p}_n).$$

<u>Proof</u>: (1) $\sigma^*[(g)^{m-1} \cdot h](0, y; \bar{p})$

$$\equiv [g \circ \sigma(0, y)]^{m-1} \cdot h(\sigma(0, y); (\sigma_*^{-1})_0^i(0, y) \cdot \bar{p}_i, \ldots, (\sigma_*^{-1})_n^i(0, y) \cdot \bar{p}_i)$$

$$\equiv [g(0, y)]^{m-1} \cdot h(0, y; [\bar{p}_0 - \frac{\partial g}{\partial p_\mu}(0, y; \delta^0) \cdot \bar{p}_\mu]/g(0, y), \bar{p}_1, \ldots, \bar{p}_n)$$

[by part (2) of Lemma 19.2]

$$\equiv \tilde{h}(0, y; \bar{p}_0 - \frac{\partial g}{\partial p_\mu}(0, y; \delta^0) \cdot \bar{p}_\mu, \bar{p}_1, \ldots, \bar{p}_n)$$

[by (19-3)].

(2) $\sigma^*[(g(x))^{m-1} \cdot h(x; p)]$

$$\equiv \sigma^*[\tilde{h}(x; g(x) \cdot p_0, p_1, \ldots, p_n] \qquad \text{[by 19.3]}$$

$$\equiv \sigma^*[\prod_{i=1}^{s} \tilde{h}_i(x; g(x) \cdot p_0, p_1, \ldots, p_n)]$$

$$\equiv \prod_{i=1}^{s} \tilde{h}_i(\sigma(\bar{x}); (g \circ \sigma) \cdot (\sigma_*^{-1})_0^i \cdot \bar{p}_i, (\sigma_*^{-1})_1^i \bar{p}_i, \ldots, (\sigma_*^{-1})_n^i \bar{p}_i)$$

$$\equiv \prod_{i=1}^{s} \bar{h}_i(\bar{x}; \bar{p}),$$

where the last equality defines the $\bar{h}_i(\bar{x}; \bar{p})$. The analyticity of $\bar{h}_i(\bar{x}; \bar{p})$ follows from part (3) of Lemma 19.2. The required expression for $\bar{h}_i(0, y; \bar{p})$ follows from part (2) of Lemma 19.2. ∎

We can now prove the results stated in Section 7. First, observe that Proposition 7.1 follows easily from part (1) of Lemma 19.2. For Theorems II and III, recall that we assumed $g(0, y) \not\equiv 0$. Therefore, the Cauchy problem (4-1) is equivalent to the following one:

$$(19\text{-}4) \qquad [g(x)]^{m-1} \cdot a(x; D)u(x) \equiv [g(x)]^{m-1} \cdot v(x)$$

$$[g(0, y)]^{j} \cdot (D_0)^{j} u(0, y) \equiv [g(0, y)]^{j} \cdot w^{j}(y), \quad j = 0, \ldots, m-1.$$

By Lemma 17.4, $u(x)$ is a solution of (19-4) if and only if $\bar{u}(\bar{x}) \equiv u \circ \sigma(\bar{x})$ is a solution of (19-2).

Let

$$\bar{a}(\bar{x}; \bar{D}) \equiv \sigma^*[(g(x))^{m-1} \cdot a(x; D)],$$

and let $\bar{h}(\bar{x}; \bar{p})$ be the principal part of $\bar{a}(\bar{x}; \bar{D})$. We have

$$[g(x)]^{m-1} \cdot a(x; D) \equiv [g(x)]^{m}(D_0)^{m} + [g(x)]^{m-1} \cdot b(x; D)$$

for some operator $b(x; D)$ of degree $\leq m-1$ in D_0. By the linearity of $\sigma *$, and Lemma 19.5, this implies that $\overline{a}(\overline{x}; \overline{D})$ is analytic on a neighborhood of 0 in $\overline{\mathbb{C}}^{n+1}$.

Part (3) of Lemma 17.3 and part (1) of Lemma 19.6 imply that

$$(19\text{-}5) \qquad \overline{h}(0, y; \overline{p}) \equiv \tilde{h}(0, y; \overline{p}_0 - \frac{\partial g}{\partial p_\mu}(0, y; \delta^0)\overline{p}_\mu, \overline{p}_1, \ldots, \overline{p}_n) .$$

Hence, $\overline{h}(0, y; \delta^0) \equiv \tilde{h}(0, y; \delta^0) \equiv 1$, so $\mathbb{C}^n$ has no characteristic points for $\overline{a}(\overline{x}; \overline{D})$.

By part (2) of Lemma 19.4, we can write

$$\sigma *[(g(0, y))^j \cdot (D_0^j)] \equiv (\overline{D}_0)^j - \sum_{i=0}^{j-1} b_i^j(y; \overline{D}_1, \ldots, \overline{D}_n)(\overline{D}_0)^i$$

for some analytic operators b_i^j on a neighborhood of 0 in $\mathbb{C}^n$. Letting $b_j^j(y; \overline{D}_1, \ldots, \overline{D}_n) \equiv 1$, we can rewrite (19-2) in the following form:

$$(19\text{-}6) \qquad \overline{a}(\overline{x}; \overline{D})\overline{u}(\overline{x}) \equiv v \circ \sigma(\overline{x})$$

$$(\overline{D}_0)^j \overline{u}(0, y) \equiv \sum_{i=0}^{j} b_i^j(y, \overline{D}_1, \ldots, \overline{D}_n)[(g(0, y))^i \cdot w^i(y)]$$

for $j = 0, \ldots, m-1$.

Under the hypotheses of Theorem III, the initial data of the Cauchy problem (19-6) are analytic. Theorem III therefore follows from the Cauchy-Kowalewski Theorem.

Now assume the hypotheses of Theorem II. Part (a) of Lemma 19.6 shows that

$$\overline{h}(\overline{x};\overline{p}) \equiv [\overline{h}_1(\overline{x};\overline{p})]^{k_1} \dots [\overline{h}_s(\overline{x};\overline{p})]^{k_s}$$

and

$$(19\text{-}7) \qquad \overline{h}_i(0,y;\overline{p}) \equiv \tilde{h}_i(0,y;\overline{p}_0 - \frac{\partial g}{\partial p_\mu}(0,y;\delta^0)\overline{p}_\mu, \overline{p}_1, \dots, \overline{p}_n).$$

Let $\tau = \overline{\alpha}_i^{(j)}(y)$ be the roots of $\overline{h}(0,y;\tau,1,0,\dots,0)$, and let $\tau = \tilde{\alpha}_i^{(j)}(y)$ be the roots of $\tilde{h}(0,y;\tau,1,0,\dots,0)$. Then (19-7) shows that

$$\overline{\alpha}_i^{(j)}(y) \equiv \tilde{\alpha}_i^{(j)}(y) + \frac{\partial g}{\partial p_1}(0,y;\delta^0).$$

Since the $\tilde{\alpha}_i^{(j)}(y)$ are all distinct numbers, for each y in some neighborhood of 0, the same is true of the $\overline{\alpha}_i^{(j)}(y)$. Hence,

$\overline{h}(\overline{x}; \overline{p})$ has constant multiplicity at 0 in the direction of $\mathbb{C}^{n-1}$. Therefore, the Cauchy problem (19-6) satisfies the hypotheses of Theorem I. Theorem II follows immediately from the application of Theorem I to (19-6).

APPENDIX

Proof of Lemma 12.4

The proof of Lemma 12.4 requires simple extensions of some results of Mizohata [8]. We will sketch their proofs, and refer the reader to that paper for the details. We therefore adopt the notation and numbering of results used in Section 5 of [8].

We let x now denote an element of $\mathbb{C}^n$ and (x, t) an element of $\mathbb{C}^{n+1}$. We let

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} a_i(x, t) \frac{\partial}{\partial x_i} + b(x, t),$$

where a_i and b are as in Section 12. We also let f, $\mathcal{O}$, Γ, U and η be as in Section 12, satisfying assumptions (1) - (4).

By assumption (3), for sufficiently large γ, γ_0 and $1/\rho$ we have

$$(5.1) \qquad |D^{\nu}_{x,t}\, a_i(x, t)| \leq \frac{(|\nu|-1)!}{(3\rho)^{|\nu|-1}}\, \gamma, \qquad |\nu| \geq 1$$

$$|a_i(x, t)| \leq \gamma_0$$

$$|D^{\nu}_{x,t}\, b(x, t)| \leq \frac{|\nu|!}{(3\rho)^{|\nu|}}\, \gamma, \qquad |\nu| \geq 0$$

for all $(x, t) \in U$.

Assuming (5.1), we now consider the solution $u(x, t)$ of the Cauchy problem

$$(5.2) \qquad L[u] = f$$

$$u(x, 0) = 0.$$

We then have the following result. (Note that this is just Mizohata's Lemma 2 with $\|t\|$ substituted for t in the bounds.)

Lemma 2: Assume

$$(5.3) \qquad |D_x^\nu f(x, t)| \leq \frac{(r+|\nu|)!}{\rho^{|\nu|}} \exp(\gamma\|t\|)K(\|t\|)^{r+|\nu|} A,$$

for all $(x, t) \in U$ and $\nu \in \mathbb{N}^n$, for some $r \geq 1$. Then

$$(5.4) \qquad |D_x^\nu u(x, t)| \leq \frac{2(r+|\nu|-1)!}{\rho^{|\nu|}} \exp(\gamma\|t\|)K(\|t\|)^{r+|\nu|} A/\gamma n,$$

for all $(x, t) \in U$, where $K(\|t\|) = \exp(\gamma n\|t\|)(1 + \gamma n\|t\|)$.

Proof: Let $(x, t) = \Gamma(\bar{x}, t)$. Then (12-4), with $g = 0$, can be written as

(A-1) $$u(x,t) = \int_0^t f(\bar{x}, s) \cdot \exp[\int_t^s b(\bar{x}, r)dr]ds,$$

where the integration is along the path η_θ, with $t = \|t\|e^{i\theta}$.

Using (5.1) and the assumption that $|\frac{d\eta_\theta}{d\tau}(\tau)| \leq 1$, we get

$$|\int_t^s b(\bar{x}, r)dr| \leq \gamma(\|t\| - \|s\|).$$

Combining this with (5.3) for $\nu = 0$, (A-1) gives

$$|u(x,t)| \leq \int_0^{\|t\|} r!\ \exp(\gamma\|s\|)K(\|s\|)^r \exp[\gamma(\|t\| - \|s\|)]\, d\|s\|$$

$$= r!\ A \exp(\gamma\|t\|) \int_0^{\|t\|} \exp(r\gamma n\|s\|)(1 + \gamma n\|s\|)^r\, d\|s\|$$

$$\leq r!\ A \exp(\gamma\|t\|)(1 + \gamma n\|t\|)^r \ \frac{\exp(r\gamma n\|t\|)}{r\gamma n}.$$

Thus, we have

(A-2) $$|u(x,t)| \leq (r-1)!\ \exp(\gamma\|t\|)K(\|t\|)^r\ A/\gamma n$$

for any $(x,t) \in U$. This proves (5.4) for $|\nu| = 0$. Note that (A-2) was obtained using (5.3) only when $|\nu| = 0$.

Now assume (5.4) holds for all ν with $|\nu| < m$. Let $|\nu| = m$ and apply D_x^ν to (5.2). As in [8], this gives

(A-3) $$L[D_x^\nu u](x,t) = F_1(x,t) + F_2(x,t)$$

where

$$|F_1(x,t)| \leq \frac{(r+m)!}{\rho^m} \exp(\gamma \|t\|) K(\|t\|)^{r+m} A,$$

$$|F_2(x,t)| \leq \frac{(r+m-1)!}{\rho^m} \exp(\gamma \|t\|) K(\|t\|)^{r+m-1} (m+1) A.$$

(We have used the fact that $\sum_{p=1}^{m} (1/3)^p < 1/2$.)

We now apply (A-2) to the Cauchy problem (A-3), since $D_x^\nu u(x,0) = 0$. More precisely, we apply it twice: for F_1 with $r+m$ substituted for r, and for F_2 with $r+m-1$ substituted for r. The superposition principle then gives

$$|D_x^\nu u(x,t)| \leq \frac{(r+m)!}{\rho^m} \exp(\gamma \|t\|) K(\|t\|)^{r+m} A \cdot [1$$

$$+ (m+1)/(r+m)(r+m-1)],$$

since $K(\|t\|) \geq 1$. Since $r, m \geq 1$, this proves (5.4). ∎

We now consider the homogeneous equation $L[u] = 0$.

<u>Lemma 3:</u> Under the above hypotheses, let $\gamma \geq 1$, let $u(x,t)$

satisfy $L[u] = 0$, and assume that

$$|D_x^{\nu} u(x, 0)| \leq \frac{(r+|\nu|)!}{\rho^{|\nu|}} A$$

for all $(x, 0) \in U$ and $\nu \in \mathbb{N}^n$, for some $r \geq 0$. Then

$$|D_x^{\nu} u(x, t)| \leq \frac{2(r+|\nu|)!}{\rho^{|\nu|}} \exp(\gamma \|t\|) K(\|t\|)^{|\nu|} A .$$

Proof: The proof is similar to that of Lemma 2, and is just outlined. Using (12-4) with $f = 0$, (5.1), $|\frac{d\eta_\theta}{d\tau}(\tau)| \leq 1$, and the hypothesis on $u(x, 0)$ for $|\nu| = 0$, we get

(A-4) $$|u(x, t)| \leq r! \frac{\exp(\gamma\|t\|)}{\gamma} A.$$

This proves the result for $|\nu| = 0$.

Now assume it is true whenever $|\nu| < m$ and let $|\nu| = m$. Applying D_x^{ν} to the relation $L[u] = 0$ gives

$$L[D_x^{\nu} u](x, t) = F_2(x, t),$$

where F_2 satisfies

$$|F_2(x, t)| \leq \frac{n(r+m)!}{\rho^m} K(\|t\|)^m (m+1)A.$$

Applying (A-4) and the hypothesis to the Cauchy problem

$$L[u_1] = 0$$

$$u_1(x, 0) = D_x^{\nu} u(x, 0)$$

gives

$$|u_1(x, t)| \leq \frac{(r+m)!}{\rho^m} \frac{\exp(\gamma\|t\|)}{\gamma} A .$$

Applying Lemma 2 to the Cauchy problem

$$L[u_2] = F_2$$

$$u_2(x, 0) = 0$$

gives

$$|u_2(x, t)| \leq \frac{(r+m-1)!}{\rho^m} \frac{\exp(\gamma\|t\|)}{\gamma} K(\|t\|)^m A(m+1).$$

Combining these inequalities gives the required bound for $D_x^{\nu} u = u_1 + u_2$. ∎

The following two propositions are proved by induction using Lemma 2 and 3. The proofs are identical to those of the corresponding results in [8], with $\|t\|$ substituted for t in the obvious places.

Proposition 3: Let $u(x, t)$ be the solution of $L[u] = f$ with $u(x, 0) = 0$. Assume that

(5.5)
$$|D_x^\nu D_t^q f(x,t)| \leq \frac{(r+q+|\nu|)!}{\rho^{q+|\nu|}} \exp(\gamma\|t\|) \cdot K(\|t\|)^{r+q+|\nu|} (\gamma n)^q A$$

for all $(x, t) \in U$, $q \in \mathbb{N}$, $\nu \in \mathbb{N}^n$, and for some $r \geq 1$. Then

(5.6)
$$|D_x^\nu D_t^q u(x,t)| \leq \frac{2(r-1+q+|\nu|)!}{\rho^{q+|\nu|}} \exp(\gamma\|t\|) \cdot K(\|t\|)^{r+q+|\nu|} (\gamma n)^q A,$$

for all $(x, t) \in U$, $q \in \mathbb{N}$, $\nu \in \mathbb{N}^n$, where γ and ρ satisfy the following condition in addition to (5.1):

(5.7)
$$\gamma \geq \min(6\,\gamma_0, 27); \quad \rho \leq 1/18.$$

Proposition 4: Let $u(x, t)$ satisfy $L[u] = 0$, and

$$(5.17) \qquad |D_x^{\nu} u(x, 0)| \leq \frac{(r+|\nu|)!}{\rho^{|\nu|}} A$$

for all $(x, 0) \in U$, $\nu \in \mathbb{N}^n$, and for some $r \geq 0$. Then

$$(5.18) \qquad |D_x^{\nu} D_t^q u(x, t)| \leq 2 \frac{(r+q+|\nu|)!}{\rho^{q+|\nu|}} \exp(\gamma\|t\|) \cdot K(\|t\|)^{r+q+|\nu|}(\gamma n)^q A$$

for all $(x, t) \in U$, $q \in \mathbb{N}$ and $\nu \in \mathbb{N}^n$, where γ and ρ are assumed to satisfy (5.7) of Proposition 3.

The proof of Lemma 12.4 is now easy. Returning to the notation of Section 12, we apply Proposition 3 to the Cauchy problem

$$\mathcal{L}[u_1] \equiv f$$

$$u_1(0, y) \equiv 0,$$

and Proposition 4 to

$$\mathcal{L}[u_2] \equiv 0$$

$$u_2(0, y) \equiv g(y) .$$

The required bound on $u(x) \equiv u_1(x) + u_2(x)$ then follows immediately by just applying the definitions of E_T and $E(r, s)$ made in Section 11.

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