

A Simple and Approximately Optimal Mechanism for an Additive Buyer

Moshe Babaioff*, Nicole Immorlica†, Brendan Lucier‡ and S. Matthew Weinberg§

Abstract

We consider a monopolist seller with n heterogeneous items, facing a single buyer. The buyer has a value for each item drawn independently according to (non-identical) distributions, and his value for a set of items is additive. The seller aims to maximize his revenue. It is known that an optimal mechanism in this setting may be quite complex, requiring randomization [19] and menus of infinite size [15]. Hart and Nisan [17] have initiated a study of two very simple pricing schemes for this setting: item pricing, in which each item is priced at its monopoly reserve; and bundle pricing, in which the entire set of items is priced and sold as one bundle. Hart and Nisan [17] have shown that neither scheme can guarantee more than a vanishingly small fraction of the optimal revenue. In sharp contrast, we show that for any distributions, the *better* of item and bundle pricing is a constant-factor approximation to the optimal revenue. We further discuss extensions to multiple buyers and to valuations that are correlated across items.

1 Introduction

A monopolist seller has a collection of n items to sell. How should he sell the items to maximize revenue given that the buyers are strategic? When there is only a single item for sale, and a single buyer with value drawn from a distribution F , Myerson [23] shows that the optimal sale protocol is straightforward: the seller should post a fixed take-it-or-leave-it price p chosen to maximize $p(1 - F(p))$, the expected revenue. The optimality of this simple auction format extends to the case of multiple buyers, as well.¹ Despite the simplicity of the single-item case, extending this solution to handle multiple items remains the primary open challenge in mechanism design. While recent work in the computer science literature has made progress on this front [2, 3, 5, 6, 8, 9, 10, 13, 15, 17, 21], it is still the case that very little is known about optimal multi-item auctions, and what *is* known lacks the simplicity of Myerson’s single-item auction.

Consider even the simplest multi-item scenario [17]: there is a single buyer² with item values drawn independently from distributions D_1, \dots, D_n , and whose value for a set of items is additive. Even when there are only two items for sale, it is known that the revenue-optimal mechanism may involve randomization [19], even to the extent of offering the buyer a choice among infinitely many lotteries [15, 18]. This is troubling not only from the perspective of analyzing optimal mechanisms, but also from the point of view of their usefulness. For an auction to be useful in practice, it should be simple to describe and transparent in its execution. Indeed,

*Microsoft Research, moshe@microsoft.com

†Microsoft Research, nicimm@microsoft.com

‡Microsoft Research, brlucier@microsoft.com

§MIT, smweinberg@csail.mit.edu. Supported by a Microsoft Graduate Research Fellowship and NSF CCF-6923736.

¹This assumes regularity of the value distributions and that the buyers’ values are drawn independently.

²Note that if the seller has unlimited copies of each item for sale, then an auction for a single buyer directly extends to the case of multiple buyers.

Myerson’s single-item auction is exciting not only for its optimality, but also its practicality.³ The danger, then, is that revenue-optimal but complex mechanisms for multiple items may share the fate of other mathematically optimal designs, such as the Vickrey-Clarke-Groves mechanism, which are very rarely used in practice [4]. It is therefore crucial to pair the study of revenue optimization with an exploration of the power of simple auctions. In other words, *what is the relative strength of simple versus complex mechanisms?*

The above question was posed in general by Hartline and Roughgarden [20], and by Hart and Nisan specifically for the setting of a single additive buyer [17]. They proposed the following suggestion for a simple multi-item auction: sell each item separately, posting a fixed price on each one. The optimal price to set on item i is then $\arg \max_p p(1 - D_i(p))$, mirroring the single-item scenario. At first glance, it appears that perhaps this simple approach should be optimal: the buyer’s value for each item is sampled independently, and her value for item i doesn’t depend at all on what other items she receives due to additivity. There is absolutely no interaction between the items at all from the buyer’s perspective, so why not sell the items separately? Somewhat counter-intuitive, it turns out that this mechanism need not achieve the optimal revenue. For example, suppose that there are n items, and that the buyer’s value for each item is distributed uniformly on $[0, 1]$. Then the optimal price to set on a single item is $\arg \max_p p(1 - p) = 1/2$, with a per-item revenue of $1/4$ and hence a total revenue of $n/4$. However, there is a different and equally straightforward mechanism that performs much better: offer only the set of *all* items at a take-it-or-leave-it price of $n(\frac{1}{2} - \epsilon)$ for some small $\epsilon > 0$. As n grows large, the probability that the sum of item values exceeds this price approaches 1, and hence the buyer is almost certain to buy. This leads to a revenue slightly less than $n/2$, a significant improvement over $n/4$. Hart and Nisan [17] show how to modify this example to exhibit a gap of $\Omega(\log(n))$ by replacing the uniform distribution with an Equal-Revenue distribution.⁴

What is going on in this example? The inherent problem is that the buyer’s value for the set of all items concentrates tightly around its expectation. This is potentially helpful for revenue generation, but the strategy of selling items separately cannot exploit this property. On the other hand, the mechanism designed to target such concentration (selling only the grand bundle at a fixed price) does very poorly in settings where concentration doesn’t occur; Hart and Nisan show that this grand-bundle mechanism achieves only an $\Omega(n)$ approximation to the optimal revenue in general. We must conclude that neither of these two simple mechanisms approximate the optimal revenue to within a constant factor.

Our main result is that the *maximum* of the revenue generated by these two approaches — either selling all items separately or selling only the grand bundle — is a constant-factor approximation to the optimal revenue. In other words, for any distribution of buyer values, either selling items separately approximates the optimal revenue to within a constant factor, or else bundling all items together does. Since a good approximation to the expected revenue of each approach can be computed in polynomial time given an appropriate access to the distribution (see Appendix G for a discussion of this claim), our results furthermore imply the first polytime constant-factor approximation mechanism for the case of an additive buyer with independently (and non-identically) distributed values, even without the restriction of simplicity.⁵ Furthermore, prior to our work, it was not even known if *any* deterministic mechanism could achieve

³This simplicity again assumes regularity and independence.

⁴The Equal-Revenue distribution has CDF $F(x) = 0$ for $x \leq 1$, and $F(x) = 1 - 1/x$ for $x \geq 1$.

⁵When the distributions are identical, and furthermore satisfy the Monotone Hazard Rate condition, [16] provides a PTAS. However, other recent results based on linear programming formulations ([1, 3, 2, 5, 8, 9, 10, 11]) all run in time polynomial in the support of D . In many correlated settings, this is the right runtime to shoot for, or the best one could hope for. But in our independent setting, this runtime will be exponential in n when ideally we would like to run in time polynomial in n . We show that if we have meaningful access to the distributions in a way that allows us to compute the optimal per-item reserves efficiently, then our mechanism runs in polynomial time.

a constant-factor approximation to the optimal mechanism, even without regard for simplicity or computational efficiency.

Main Result (Informal). *In any market with a single additive buyer and arbitrary independent item value distributions, either selling every item separately or selling all items together as a grand bundle generates at least a constant fraction of the optimal revenue.*

Our result nicely complements an active research area aimed at characterizing distributions and valuations in which simple mechanisms are *precisely* optimal [2, 17, 24, 25]. In contrast to that literature, we show that a maximum over simple mechanisms is *approximately* optimal, for *arbitrary* distributions and *additive* valuations. Our result also echoes a similar line of investigation for markets with unit-demand valuations in which a buyer’s value for a set of items is his maximum value for an item in the set. In this setting, it is known [12, 13, 14] that selling items separately achieves a constant approximation to the optimal revenue. Our result illustrates that a similar approximation can be achieved for additive buyers, provided that we also consider selling all items together as a grand bundle.

To obtain some intuition into our result, recall the example above with n items and uniformly-distributed values. This example illustrates that selling all items separately may be a poor choice when the value for the grand bundle concentrates around its expectation. What we show is that, in fact, this is the *only* scenario in which selling all items separately is a poor choice. We prove that if the total value for all items does *not* concentrate, then selling separately must generate a constant fraction of optimal revenue.

Our argument makes use of a core-tail decomposition technique introduced by Li and Yao [22] to study the revenue of selling items separately. Roughly speaking, the idea is to split the support of each item’s value distribution into a “tail” (those values that are sufficiently large), and a “core” (the remainder). One then attributes the revenue of the optimal mechanism to the revenue extracted from values in the tail, plus the expected sum of values in the core. To bound the optimal revenue, it then suffices to bound each of these two quantities separately. Li and Yao define the tail of a distribution so that each value is in the tail with probability at most $1/n$; they use this to prove that selling all items separately obtains a logarithmic approximation to the optimal revenue (which is tight).

We apply a similar approach, but we define the boundary between core and tail in a different way. We aim to strike a balance between two opposing goals: we want the boundaries to be high enough that the probability of being in the tail is low, which will imply that the revenue from the tail is small relative to selling items separately. At the same time, we want values in the core to be small enough that, subject to their sum being large, the sum must necessarily concentrate around its expectation (which would imply that bundling all items together achieves good revenue). To meet these two goals, we design thresholds that are adapted to the revenue contributions of different items, which makes the core smaller (relative to non-adaptive thresholds) when the value distributions are highly asymmetric. This gives us the extra flexibility needed to derive a constant-factor approximation.

We apply the same methodology to prove that when there are many buyers (with valuations that are not necessarily samples from identical distributions), selling all items separately yields an $O(\log(n))$ approximation to the optimal mechanism. This bound is asymptotically tight, as Hart and Nisan have presented a lower bound that matches this for just a single buyer. Prior to our work, no non-trivial bounds were known on the revenue of selling separately to many buyers, or even on the revenue on *any* class of mechanisms. Furthermore, the observation that selling separately fails only under concentration has implications in this setting as well: we further show that unless the maximum attainable welfare (of all buyers together) concentrates, that selling items separately again obtains a constant-factor approximation. However, with many buyers the concentration of welfare does *not* imply that selling the grand bundle together obtains a constant-factor approximation. Indeed, unlike in the single-buyer case, one cannot

improve the approximation ratio by using bundling: we prove that the $\Omega(\log(n))$ lower bound applies against the better of selling separately and together as well. This realization motivates our first open problem:

Open Problem 1. *Is there a “simple,” approximately optimal mechanism for many additive buyers with independent values?*

In attempt to make progress on this problem, we turn to a subclass of deterministic mechanisms that we call “partition mechanisms.” A partition mechanism first partitions the items into disjoint bundles, then sells each bundle separately. This natural class of mechanisms clearly generalizes both selling separately and selling together, so we study the performance of the optimal mechanism in this class relative to that of others. On this front, we show that unfortunately the revenue of the optimal mechanism for many independent buyers can still be an $\Omega(\log n)$ factor larger than that of the optimal partition mechanism, and further that revenue of the optimal partition mechanism can be an $\Omega(\log n)$ factor larger than the better of selling separately and together.⁶

Finally, we study the performance of selling separately and together against partition mechanisms for a single buyer whose values for the items may be arbitrarily correlated. While neither class of mechanisms can guarantee any finite factor of the optimal revenue ([7, 17]), the question remains as to whether simple mechanisms can approximate more complex (though still suboptimal) mechanisms in the presence of correlation. To this end, we prove that selling items separately obtains an $O(\log n)$ -approximation the optimal obtainable revenue by a partition mechanism, and that this is tight. In fact, we show a gap of $\Omega(\log n)$ between the better of selling separately and together versus the optimal partition mechanism. We include several tables in Appendix A displaying the relative power of the various classes of mechanisms studied in this paper, noting here that as of our work, all upper and lower bounds are (asymptotically) matching.

Our paper leaves several natural open problems for future work. The first was already stated and concerns extending our results to many buyers. A second problem concerns extending our results beyond additive valuations. As for both unit-demand and additive valuations a constant-factor approximation mechanism is now known, one could naturally ask if such a result is also achievable for valuations that generalize both unit-demand and additive. One potential instantiation is a buyer with a k -demand valuation; i.e., additive, but wants at most k items. A significantly more challenging instantiation is the class of gross-substitute valuations.

Open Problem 2. *Is there a “simple,” approximately optimal mechanism for single buyer with a k -demand valuation? With a gross-substitute valuation?*

Finally, a third problem concerns extending our results to settings with mild (but not arbitrary) correlation. This approach was fruitful in [14] for the “common base-value” model.⁷

Open Problem 3. *Is there a “simple,” approximately optimal mechanism for a single additive buyer whose value for n items is sampled from a common base-value distribution? What about other models of limited correlation?*

2 Preliminaries

The setting we consider is that of a single monopolist seller with n heterogeneous and indivisible items for sale to m additive, risk-neutral, quasi-linear consumers (buyers). That is, each

⁶Clearly, no example can exhibit both gaps simultaneously as selling separately achieves an $O(\log n)$ -approximation to the optimal revenue.

⁷In the common base-value model, the buyer has $n + 1$ distributions D_0, \dots, D_n , and samples v_i from each D_i . Her value for item $i \in \{1, \dots, n\}$ is then $v_0 + v_i$, and v_0 is called the “base-value.”

consumer j has a value v_{ij} for item i . While our main results are for the setting of a single buyer, we will define our setting more generally; this will be useful when discussing extensions. If a randomized outcome awards consumer j item i with probability π_{ij} and charges him a price p_j in expectation, then his utility for this outcome is $\sum_i v_{ij}\pi_{ij} - p_j$. Each value v_{ij} is sampled independently from a known distribution D_{ij} . We make no assumptions on D_{ij} whatsoever. We refer to D as the joint mn -dimensional distribution over all consumers' values for all items, D_i as the m -dimensional distribution over all consumers' values for item i . Furthermore, we denote by \vec{v} a random sample from D , \vec{v}_i a random sample from D_i . We also denote the maximum value for item i as $v_i^* = \max_j \{v_{ij}\}$.

We are interested in analyzing mechanisms at Bayes-Nash equilibrium of buyer behavior, with an eye toward maximizing revenue at equilibrium. By the revelation principle, we can restrict attention to mechanisms that are Bayesian Incentive Compatible (i.e., truthful).⁸ As usual, we also impose the individual rationality constraint, saying that every buyer's utility is non-negative when truthful.

We use the following terminology to discuss the revenue obtainable by various types of mechanisms, where the first three are taken from [17].

- $\text{REV}(D)$: The optimal revenue obtained by any (possibly randomized) truthful mechanism when the consumer profile is drawn from D .
- $\text{SREV}(D)$: The optimal revenue obtained by auctioning items separately when the consumer profile is drawn from D . That is, the revenue obtained by running Myerson's optimal auction separately for each item.
- $\text{BREV}(D)$: The optimal revenue obtained by auctioning the grand bundle when the consumer profile is drawn from D . That is, the revenue obtained by running Myerson's optimal auction when treating the grand bundle as a single item.
- $\text{PREV}(D)$: The optimal revenue obtained by any *partition* mechanism when the consumer profile is drawn from D . That is, the maximal revenue obtained by first partitioning the items into disjoint bundles, and then running Myerson's optimal auction separately for each bundle, treating each bundle as a single item.

Given a distribution D over profiles, we will often consider the welfare $\sum_i v_i^*$ of a consumer profile \vec{v} drawn from D . We will write $\text{VAL}(D)$ for the expected welfare, so that $\text{VAL}(D) = \mathbb{E}_{\vec{v} \sim D} [\sum_i v_i^*]$. We will also write $\text{var}(D) = \text{var}_{\vec{v} \sim D} (\sum_i v_i^*)$ for the variance of the welfare.

We will make use of some results from [17] that provide useful bounds on $\text{REV}(D)$. We include proofs in Appendix B for completeness. Lemma 1 is stated and proved directly in [17]. Lemma 2 is not directly stated nor proved, but is similar to an implicit result from [17].

In the lemma below, we think of D and D' as being distributions over values for disjoint sets of items, for the same set of m consumers. The distribution $D \times D'$ then draws values for those two sets of items, independently, from D and D' respectively.

Lemma 1. ([17]) $\text{REV}(D \times D') \leq \text{VAL}(D) + \text{REV}(D')$.

The next result establishes a weak bound on $\text{REV}(D)$ with respect to $\text{SREV}(D)$.

Lemma 2. $\text{REV}(D) \leq nm\text{SREV}(D)$.

⁸As it turns out, all of the mechanisms we describe will also satisfy the stronger property of dominant strategy truthfulness.

3 The Core Decomposition

We make use of an idea developed by Li and Yao [22] called the “core” of a value distribution for a single consumer. In order to obtain our stronger results for a single consumer and also extend to many consumers, we define the core differently but in the same spirit. The idea is to separate each m -dimensional value distribution for each item into the core and the tail, the tail being the part where some consumer has an unusually high value for the item. Then the core of the entire nm -dimensional distribution is the product of all the cores, and the tail is everything else.

3.1 Defining the Core and Prior Results

Below we formalize the notion of the core. We introduce some notation that will be used throughout the paper. By the “null” distribution, we mean a distribution whose product with any other distribution is also a null distribution, and that outputs \perp with probability 1.

- r_i : The optimal revenue obtainable by selling just item i (using Myerson’s optimal auction).
- r : $\sum_i r_i$. The same as $SRev(D)$ but cleaner to write in formulas.
- t_i : A profile of parameters, one per item, to define the separation between the core and tail of distribution F_i . We will think of t_i as a multiplier applied to r_i . The core for item i will be supported on the interval $[0, t_i r_i]$, and the tail for item i will be supported on $(t_i r_i, \infty)$. Different results throughout the paper will specify different choices for t_i .
- p_i : $Pr[v_i^* > t_i r_i]$, the probability that the highest value on item i lies in the tail. Note that this may be 0.
- D_i^C : The core of D_i , the conditional distribution of \vec{v}_i conditioned on $v_i^* \leq t_i r_i$. Note that this may be the null distribution if $p_i = 1$.
- D_i^T : The tail of D_i , the conditional distribution of \vec{v}_i conditioned on $v_i^* > t_i r_i$. Note that this may be the null distribution if $p_i = 0$.
- A : Throughout our notation, we will use A to represent a subset of items. We often think of A as the items whose values lie in the tail of their respective distributions.
- D_A^T : A is a subset of items, and D_A^T is a product distribution equal to $\times_{i \in A} D_i^T$.
- D_A^C : A is a subset of items, and D_A^C is a product distribution equal to $\times_{i \notin A} D_i^C$.
- D_A : $D_A^C \times D_A^T$. Note that this product is taken over the tail of items in A and the core of items not in A . In other words, D_A is the distribution D , conditioned on $v_i^* > t_i r_i$ if $i \in A$ and conditioned on $v_i^* \leq t_i r_i$ if $i \notin A$.
- p_A : $Pr[\vec{v} \in support(D_A)]$. This is equal to $(\prod_{i \in A} p_i)(\prod_{i \notin A} (1 - p_i))$.

Before stating our core decomposition lemma, we present some known results about the core. The lemmas below were either stated explicitly in [22] or [17], or use ideas from one of those papers. We put a citation in the statement of such lemmas, but include all proofs in Appendix C for completeness.

Lemma 3. ([22]) $p_i \leq 1/t_i$ for all i .

Lemma 4. ([22]) $REV(D_i^C) \leq r_i$ and $REV(D_i^T) \leq r_i/p_i$.

Lemma 5. ([17]) $REV(D) \leq \sum_A p_A REV(D_A)$.

3.2 The Core Decomposition Lemma

In this section we state our Core Decomposition Lemma, which relates the optimal revenue from a distribution D to the revenue and welfare that can be extracted from the tail and core of D . This result is similar in spirit to the core lemma of [22].

Our first result, Lemma 6, is our main decomposition lemma. The lemma states that the optimal revenue from distribution D can be split into a contribution from the core of D and a contribution from the tail of D . One might hope for a bound of the form “the optimal revenue from D is at most the optimal revenue from the tail plus the optimal revenue from the core.” Indeed, such a bound is attainable for a single buyer [22], but is problematic for many buyers, see Section 4.4 and Appendix 3 in [17] for a discussion. We will therefore settle for a weaker bound: the optimal revenue from the tail plus the *expected welfare* from the core. We also note that the approach of Li and Yao eventually upper bounds the optimal revenue of the core with the expected welfare anyway.

Lemma 6 (Core Decomposition). $\text{REV}(D) \leq \text{VAL}(D_\emptyset^C) + \sum_A p_A \text{REV}(D_A^T)$

Proof. By Lemma 1,

$$\text{REV}(D_A) \leq \text{VAL}(D_A^C) + \text{REV}(D_A^T)$$

for all A . Also, since $\text{VAL}(D_A^C)$ is the expected sum of values for items *not* in A , we have

$$\text{VAL}(D_A^C) \leq \text{VAL}(D_\emptyset^C).$$

By Lemma 5,

$$\begin{aligned} \text{REV}(D) &\leq \sum_A p_A \text{REV}(D_A) \\ &\leq \sum_A p_A (\text{VAL}(D_A^C) + \text{REV}(D_A^T)) \\ &\leq \left(\sum_A p_A \right) \text{VAL}(D_\emptyset^C) + \sum_A p_A \text{REV}(D_A^T). \end{aligned}$$

As $\sum_A p_A = 1$ the desired result follows. \square

4 Revenue Bounds for a Single Buyer

In this section we focus on the case of a single buyer, $m = 1$. We will work toward proving our main result, which is that $\max\{\text{SREV}(D), \text{BREV}(D)\}$ is a constant-factor approximation to $\text{REV}(D)$ in this setting. Our argument will make use of the core decomposition, described in the previous section. We will begin with a simpler result that illustrates our techniques: that $\text{REV}(D)$ is at most $(\ln n + 3)$ times $\text{SREV}(D)$. A logarithmic approximation was already established in [22]; we obtain a slightly tighter bound, but the primary purpose of presenting this result is as a warm-up to introduce our techniques and those of [22]. We will then show how this bound can be improved to a constant by considering the maximum of $\text{SREV}(D)$ and $\text{BREV}(D)$.

4.1 Warm-up: $(\ln n + 3)\text{SREV} \geq \text{REV}$

We first give a simple application of our approach to provide a bound on SREV vs. REV , which is slightly improved relative to the bound obtained in [22].

Theorem 1. *For a single buyer, and any $c > 0$, $(2 + 1/c + \ln c + \ln n)\text{SREV}(D) \geq \text{REV}(D)$. This is minimized at $c = 1$, yielding $(\ln n + 3)\text{SREV}(D) \geq \text{REV}(D)$.*

The idea of the proof is to consider the core decomposition of D , choosing $t_i = cn$ for each item i . By the Core Decomposition Lemma (Lemma 6), Theorem 1 follows if we can bound the optimal revenue from the tail and the expected welfare from the core, given this choice of $\{t_i\}_i$.

We begin with Proposition 1, which effectively shows that for constant c , the revenue from the tail is at most a constant times $\text{SREV}(D)$. The intuition behind this result is that each item i lies in the tail with probability $p_i \leq 1/t_i = 1/cn$, and hence a large fraction of the time there will be at most a single item whose value lies in the tail. In this case, the revenue from the values in the tail is certainly no more than $\text{SREV}(D)$, since the optimal mechanism can do no better than setting the optimal price for the single item present. To bound the revenue contribution when many values lie in the tail, the relatively weak bound in Lemma 2 will suffice.

Proposition 1. *For a single buyer, and any $c > 0$, if $t_i = cn$ for all i , then $\sum_A p_A \text{REV}(D_A^T) \leq (1 + 1/c)\text{SREV}(D)$.*

Proof. By Lemma 2 and Lemma 4, $\text{REV}(D_A^T) \leq |A|\text{SREV}(D_A^T) \leq \sum_{i \in A} |A|r_i/p_i$. Therefore, we may rewrite the sum by first summing over item i , and then sets A containing i , obtaining:

$$\sum_A p_A \text{REV}(D_A^T) \leq \sum_i \sum_{A \ni i} |A| p_A r_i / p_i.$$

We now wish to interpret the term $\sum_{A \ni i} |A| p_A / p_i$. Observe that p_A / p_i is exactly the probability that the set A of items are in the tail, conditioned on i being in the tail, and $|A|$ is just the size of A . Summing over all $A \ni i$ therefore yields the expected size of the set of items in the tail, conditioned on i being on the tail.⁹ Clearly this expectation is just $1 + \sum_{j \neq i} p_j$, which is at most $1 + 1/c$ by Lemma 3. As we have just observed that $\sum_{A \ni i} |A| p_A / p_i \leq 1 + 1/c$, we have now shown that $\sum_A p_A \text{REV}(D_A^T) \leq \sum_i (1 + 1/c)r_i$, which is exactly $(1 + 1/c)\text{SREV}(D)$. \square

Having established a bound on the revenue of the tail, we turn to the welfare of the core. For this, we use the definition of $r_i = \text{SREV}(D_i)$ to directly bound $\Pr[v_i > x]$ for all x , and then take an expectation over the range of the core.

Proposition 2. *For a single buyer, and any $c > 0$, if $t_i = cn$ for all i , then $(1 + \ln c + \ln n)\text{SREV}(D) \geq \text{VAL}(D_\emptyset^C)$.*

Proof. Note that $\text{VAL}(D_\emptyset^C) = \sum_i \text{VAL}(D_i^C) \leq \sum_i \int_0^{cnr_i} \Pr[v_i > x] dx$. The last inequality would be equality if we replaced v_i with a random variable drawn from D_i^C , but since v_i stochastically dominates such a random variable, we get an inequality instead. As the optimal revenue of D_i is r_i , this means that $\Pr[v_i > x] \leq \min\{1, r_i/x\}$. So we have

$$\text{VAL}(D_i^C) \leq \int_0^{r_i} dx + \int_{r_i}^{cnr_i} r_i/x dx = r_i + r_i(\ln(cnr_i) - \ln(r_i)) = r_i(1 + \ln n + \ln c)$$

Summing this guarantee over all i yields the proposition. \square

Combining Propositions 1 and 2 with Lemma 6 yields Theorem 1.

4.2 Main Result: $6 \cdot \max\{\text{SREV}, \text{BREV}\} \geq \text{REV}$

In this section we prove our main result, showing that the best of selling items separately and bundling all of them together is a constant-factor approximation to the optimal mechanism. The proof will follow a similar skeleton to that of Section 4.1, by proving propositions similar to

⁹This observation is due to Aviad Rubinfeld, and we thank him for allowing us to include it. An earlier version of this paper presented a $(\ln n + 5)$ -approximation in Theorem 1 and a 7.5-approximation in Theorem 2. This observation improved those factors to $(\ln n + 3)$ and 6, respectively.

Propositions 1 and 2. The notable difference is that we will need to be more careful in defining the core.

When all D_i are identical, the approach in Section 4.1 (setting each $t_i = cn$) can be leveraged to yield the bound $O(1) \cdot \text{BREV} \geq \text{REV}$ ([22]), but fails in the case that a small number k of items contributes the majority of the optimal revenue. To see the problem, note that the definition of the core depends on the number of items n , but this can be made arbitrarily large by adding extra items of negligible value. The effect is that the core is potentially larger than necessary when value distributions are asymmetric. What we need instead is for t_i to depend on the value distribution D_i . We let t_i scale inverse proportionally to r_i , so that high-revenue items are more likely to occur in the tail. This allows us to capture scenarios in which revenue comes primarily from one heavy item (by analyzing the tail), as well as instances driven by the combined contribution of many light items (by analyzing the core). Indeed, note that if we set $t_i = cr/r_i$, then the boundary between core and tail becomes $t_i r_i = cr = c\text{SREV}(D)$ for each item. This turns out to be precisely the threshold that we need to attain constant-factor approximation bounds for both the core and the tail, simultaneously.

Theorem 2. *For a single buyer, $\text{REV}(D) \leq 6 \max\{\text{SREV}(D), \text{BREV}(D)\}$.*

As in Theorem 1, our approach will be to apply the Core Decomposition Lemma (Lemma 6) with an appropriate choice of values t_i , then bound separately the revenue from the tail and the welfare from the core. As discussed above, we will make the non-uniform choice $t_i = r/r_i$ for each i .

Proposition 3. *For a single buyer, when $t_i = r/r_i$ for each i , $\sum_A p_A \text{REV}(D_A^T) \leq 2\text{SREV}(D)$.*

Proof. We begin similarly to the proof of Proposition 1, using Lemma 2 and Lemma 4 to write $\text{REV}(D_A^T) \leq |A| \text{SREV}(D_A^T) \leq \sum_{i \in A} |A| r_i / p_i$. Again, summing this over all A yields:

$$\sum_A p_A \text{REV}(D_A^T) \leq \sum_i \sum_{A \ni i} |A| p_A r_i / p_i.$$

Just like in Proposition 1, $\sum_{A \ni i} |A| p_A / p_i$ is exactly the expected number of items in the tail, conditioned on i being in the tail. It's again clear that this sum is exactly $1 + \sum_{j \neq i} p_j$. By Lemma 3, this is at most $1 + \sum_{j \neq i} 1/t_j$. By our choice of t_i , the second term is upper bounded by 1, as $t_j = r/r_j$ and $\sum_j r_j = r$. Therefore, $\sum_{A \ni i} |A| p_A / p_i \leq 2$, and $\sum_A p_A \text{REV}(D_A^T) \leq 2\text{SREV}(D)$. \square

We now turn to bounding the welfare from the core. We will use the small range of the core to derive an upper bound on the variance of its welfare. This will allow us to conclude that the welfare is highly concentrated whenever it is sufficiently large relative to $\text{SREV}(D)$. Thus, if the welfare is “small” compared to $\text{SREV}(D)$, then selling separately extracts most of the welfare (within the core); otherwise the welfare concentrates and so bundling extracts most of the welfare (within the core). The following lemma of [22] will be helpful for this approach; its proof appears in Appendix D for completeness.

Lemma 7. ([22]) *Let F be a one-dimensional distribution with optimal revenue at most c supported on $[0, tc]$. Then $\text{var}(F) \leq (2t - 1)c^2$.*

Corollary 1. *For a single buyer, and any choice of t_i , $\text{var}(D_i^C) \leq 2t_i r_i^2$.*

Proof. $\text{REV}(D_i^C) \leq r_i$, and the distribution D_i^C is supported on $[0, t_i r_i]$. Therefore, plugging into Lemma 7 (and relaxing) yields the desired bound. \square

Proposition 4. *For a single buyer, when all $t_i = r/r_i$, $\max\{\text{SREV}(D), \text{BREV}(D)\} \geq \frac{1}{4} \text{VAL}(D_\emptyset^C)$.*

Proof. There are two cases to consider. If $\text{VAL}(D_\emptyset^C) \leq 4r$, then we have that $\text{SREV}(D) = r \geq \frac{1}{4}\text{VAL}(D_\emptyset^C)$ as required.

On the other hand, if $\text{VAL}(D_\emptyset^C) \geq 4r$, then Corollary 1 tells us that $\text{var}(D_i^C) \leq 2t_i r_i^2$. Summing over all i and recalling that $t_i = r/r_i$ we get

$$\text{var}(D_\emptyset^C) = \sum_i \text{var}(D_i^C) \leq 2 \sum_i t_i r_i^2 = 2r^2.$$

So $\text{var}(D_\emptyset^C) \leq 2r^2$ and $\text{VAL}(D_\emptyset^C) \geq 4r$. By Chebyshev's inequality, we get

$$\Pr \left[\sum_i v_i \leq \frac{2}{5} \cdot \text{VAL}(D_\emptyset^C) \right] \leq \frac{2r^2}{(1 - \frac{2}{5})^2 \cdot \text{VAL}(D_\emptyset^C)^2} \\ \leq \frac{25r^2}{72r^2} = \frac{25}{72}.$$

Since $\text{BREV}(D)$ is at least the revenue obtained by setting price $\frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)$ on the grand bundle, this implies $\text{BREV}(D) \geq (\frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)) \cdot \frac{47}{72} = \frac{47}{180} \cdot \text{VAL}(D_\emptyset^C)$. As $\frac{47}{180} > \frac{1}{4}$, $\text{BREV}(D) > \frac{1}{4}\text{VAL}(D_\emptyset^C)$ as required. \square

Combining Propositions 3 and 4 with Lemma 6 yields Theorem 2. To our knowledge, the best known lower bound on $\max\{\text{SREV}, \text{BREV}\}$ vs. REV is 1.05, provided by an example in [15]. The example has two items with $D_1 = U\{1, 2\}$ and $D_2 = U\{1, 3\}$, with $\text{REV} = 2.625$, $\text{SREV} = 2.5$, and $\text{BREV} = 2.25$. It is an interesting open question to close the gap between 1.05 and 6, either by tightening our analysis or providing better lower bounds.

5 Revenue Bounds for Multiple Buyers

Here we extend our results to multiple buyers with valuations sampled independently (but not necessarily identically). We will refer to this as the *independent setting*, as the buyers' valuations are independent and furthermore each buyer's item values are also drawn independently. We first show in Theorem 3 that for the independent setting, selling items separately achieves a logarithmic (in n) approximation to the optimal revenue. We next show in Theorem 5 that like in the single buyer case, the only case in which selling items separately fails to achieve a good approximation, is the case that welfare is highly concentrated. Unfortunately, such concentration is no longer sufficient to achieve a constant approximation by selling all items together. This is so because even though the welfare is concentrated, the partition that provides such welfare can change dramatically between realizations. Indeed, in Proposition 8 we show not only that $\text{BREV}(D)$ fails to provide a constant approximation to the optimal mechanism, but even $\text{PREV}(D)$ fails, and this is so even when item values are sampled i.i.d. for all items and buyers. Finally, in Proposition 9 we show that in the independent setting, $\text{PREV}(D)$ cannot be approximated well by $\max\{\text{SREV}(D), \text{BREV}(D)\}$.

5.1 An Upper Bound: $(\ln n + 6)\text{SREV} \geq \text{REV}$

We first show that selling items separately achieves a logarithmic (in n) approximation to the optimal revenue.

Theorem 3. *For arbitrarily many buyers, in the independent setting, $(2 + 2e^{1/4} + \ln 4 + \ln n)\text{SREV}(D) \geq \text{REV}(D)$. (Note that $2 + 2e^{1/4} + \ln 4 < 6$.)*

Our proof will proceed via *amplification*. We will begin with the (awful) bound on SREV vs. REV from Lemma 2, then show in Theorem 4 how to amplify any such bound into an improved bound. We will then iterate this amplification process over and over, until we reach the desired

logarithmic approximation (which will be a fixed point of the amplification process). To prove the amplification theorem, we use an approach similar to the single-buyer analysis from Section 4.1. That is, we will apply the Core Decomposition Lemma (Lemma 6), then bound the revenue of the tail and the welfare of the core with respect to $\text{SREV}(D)$.

Theorem 4 (Amplification). *For arbitrarily many buyers in the independent setting, assume that for some $a > 1$ it holds that $an\text{SREV}(D) \geq \text{REV}(D)$. Then, for any $c \geq 1$, $(2 + 2e^{1/ca}/c + \ln c + \ln a + \ln n)\text{SREV}(D) \geq \text{REV}(D)$ as well. Setting $c = 1$ yields $(2 + 2e^{1/a} + \ln a + \ln n)\text{SREV}(D) \geq \text{REV}(D)$.*

To prove Theorem 4, we will apply the Core Decomposition Lemma (Lemma 6), using $t_i = c \cdot a \cdot n$ for each i . Theorem 4 will then follow from bounds on the revenue from the tail and the expected welfare from the core, which we establish in the following propositions.

Proposition 5. *For arbitrarily many buyers in the independent setting, if $t_i = c \cdot a \cdot n$ for all i and $an\text{SREV}(D) \geq \text{REV}(D)$, then $\sum_A p_A \text{REV}(D_A^T) \leq (1 + 2e^{1/ca}/c)\text{SREV}(D)$.*

Proof. The proof is nearly identical to that of Proposition 1. The only difference is that we start with the fact that $\text{REV}(D_A^T) \leq a|A|\text{SREV}(D_A^T)$ instead of just $|A|$, and make use of the fact that when there is only one item, $\text{SREV}(D_A^T) = \text{REV}(D_A^T)$. We include the details in Appendix E for completeness. \square

The following bound on the welfare from the core follows in a manner similar to Proposition 2. We defer its proof to Appendix E.

Proposition 6. *For arbitrarily many buyers in the independent setting, if $t_i = c \cdot a \cdot n$ for all i , then $(1 + \ln c + \ln a + \ln n)\text{SREV}(D) \geq \text{VAL}(D_\emptyset^C)$.*

Theorem 4 then follows from Propositions 5 and 6, together with Lemma 6. We now show how to prove Theorem 3 using Theorem 4.

Proof (of Theorem 3). By Lemma 2, we may apply Theorem 4 starting with $a = m$. This yields a bound of the form $a'n\text{SREV}(D) \geq \text{REV}(D)$ for some new a' . We can then apply Theorem 4 again, taking a to be this new value a' . We can iteratively apply Theorem 4 over and over until we either reach a fixed point (with respect to the value of a) or reach $a = 1$. One can verify that, for all $n \geq 2$, no $a \geq 4$ is a fixed point and that the function $f(a) = (2 + 2e^{1/a} + \ln a + \ln n)/n$ is continuous. Therefore, we can always iterate until $a \leq 4$ and then apply Theorem 4 with $a = 4$, yielding the desired bound. \square

5.2 A Concentration Result

We next present a characterization of when $\text{SREV}(D)$ is a constant-factor approximation to $\text{REV}(D)$ for the independent setting with multiple buyers. We will show (in Theorem 5, below) that this occurs unless the welfare of D is sufficiently well concentrated around its expectation.

We begin with a corollary of Theorem 3, which will be useful for our analysis.

Corollary 2. *For arbitrarily many buyers in the independent setting, $4n\text{SREV}(D) \geq \text{REV}(D)$.*

Proof. This is a direct application of Theorem 3 and noting that $6 + \ln n \leq 4n$ for all $n \geq 2$. \square

We next prove an alternative bound on the revenue from the tail of the distribution D , using a familiar choice of t_i . The proof, which closely follows that of Proposition 3, appears in Appendix E.

Proposition 7. *For arbitrarily many buyers in the independent setting, if we choose $t_i = 4r/r_i$ for all i , then $\sum_A p_A \text{REV}(D_A^T) \leq 5\text{SREV}(D)$.*

We are now ready to establish the claimed bound between SREV and REV, subject to the welfare of D not being too concentrated around its expectation.

Definition 1. We say that a one-dimensional distribution F is d -concentrated if there exists a value C such that $\Pr_{x \sim F}[|x - C| \leq C/2] \geq d$.

Theorem 5. For arbitrarily many buyers in the independent setting, and any $c \geq 4\sqrt{2}$, either $(c+5)\text{SREV}(D) \geq \text{REV}(D)$ or the welfare of D (the random variable with expectation $\text{VAL}(D)$) is $(3/4 - \frac{24}{c^2})$ -concentrated.

Proof. Let all $t_i = 4r/r_i$. Then combining Proposition 7 and Lemma 6 yields

$$5\text{SREV}(D) + \text{VAL}(D_\emptyset^C) \geq \text{REV}(D).$$

There are two cases to consider. Maybe $c\text{SREV}(D) \geq \text{VAL}(D_\emptyset^C)$. In this case, we have $(c+5)\text{SREV}(D) \geq \text{REV}(D)$.

On the other hand, maybe $\text{VAL}(D_\emptyset^C) \geq c\text{SREV}(D)$. In this case, Corollary 1 tells us that $\text{var}(D_\emptyset^C) \leq 2t_i r_i^2$. Summing over all i and recalling that $t_i = 4r/r_i$, we get

$$\text{var}(D_\emptyset^C) \leq 2 \sum_i t_i r_i^2 = 2 \sum_i (4r) r_i = 8r^2.$$

So $\text{var}(D_\emptyset^C) \leq 8r^2$ and $\text{VAL}(D_\emptyset^C) \geq cr$. By Chebyshev's inequality, we get

$$\Pr \left[\left| \sum_i v_i^* - \text{VAL}(D_\emptyset^C) \right| \geq \text{VAL}(D_\emptyset^C)/2 \right] \leq \frac{8r^2}{\text{VAL}(D_\emptyset^C)^2/4} \leq \frac{32r^2}{c^2 r^2} = \frac{32}{c^2}$$

meaning that the welfare of D_\emptyset^C is $(1 - \frac{32}{c^2})$ -concentrated. The last step is observing that \vec{v} is sampled in the support of D_\emptyset^C with probability exactly $\prod_i (1 - p_i)$. As $\sum_i p_i \leq 1/4$ and each $p_i \leq 1/4$, this is minimized when exactly one p_i is $1/4$ and the rest are 0, yielding $\prod_i (1 - p_i) = 3/4$. So with probability at least $3/4$ \vec{v} is in the support of D_\emptyset^C . When this happens, the welfare is $(1 - \frac{32}{c^2})$ concentrated. So the welfare of D is $(3/4 - \frac{24}{c^2})$ -concentrated. \square

5.3 A Lower Bound: $\text{PREV} \leq \text{REV}/\Omega(\log n)$ even for i.i.d. Item Values

We next show that there is a setting with many buyers with item valuations that are sampled i.i.d from the same distribution, for which $\text{PREV}(D)$ (and thus also $\max\{\text{SREV}(D), \text{BREV}(D)\}$) provides a poor approximation to $\text{REV}(D)$.

Proposition 8. There exists a setting with n items and many buyers, with item valuations that are sampled i.i.d from the same distribution, for which $\text{PREV}(D) \leq \text{REV}(D)/\Omega(\log n)$.

Proof. Consider a setting with n items and \sqrt{n} buyers with the following value distributions. For every item i and buyer j , the distribution $D_{i,j}$ such that the value is 0 with probability $1 - 1/\sqrt{n}$, and with the remaining probability it is sampled from a distribution F with CDF $F(x) = 1 - x^{-1}$ for $x \in [1, n^{1/8}]$ and $F(x) = 1$ for $x > n^{1/8}$ (an Equal-Revenue distribution with all mass above $n^{1/8}$ moved to an atom at $n^{1/8}$). To prove the claim we show in Lemma 11 in Appendix E that $\text{REV}(D) \in \Omega(n \log n)$ while $\text{PREV}(D) \in O(n)$ (actually, since $\text{SREV} \in \Omega(n)$ it holds that $\text{PREV}(D) \in \Theta(n)$). \square

5.4 A Lower Bound: $\max\{\text{SREV}, \text{BREV}\} \leq \text{PREV}/\Omega(\log n)$

We next show that there is a setting with many buyers with item valuations that are sampled independently (but not identically), for which $\max\{\text{SREV}(D), \text{BREV}(D)\}$ provides a poor approximation to $\text{PREV}(D)$.

Proposition 9. *There exists a independent setting with n items and many buyers for which $\max\{\text{SREV}(D), \text{BREV}(D)\} \leq \text{PREV}(D)/\Omega(\log n)$.*

Proof. Fix n such that \sqrt{n} is an integer. Consider a setting with \sqrt{n} buyers, and a partition the items to \sqrt{n} disjoint sets of size \sqrt{n} . Buyer k has value 0 for every item that is not in the k -th set of items, and for item in that set his value is sampled independently from an Equal-Revenue distribution.

Clearly, $\text{SREV}(D) = n$. $\text{BREV}(D)$ is the same as the revenue that $\text{BREV}(D)$ can get in a setting with \sqrt{n} buyers and only \sqrt{n} items for which each item value is sampled i.i.d. from an Equal-Revenue distribution. That revenue is $O(\sqrt{n} \log \sqrt{n})$. We conclude that $\max\{\text{SREV}(D), \text{BREV}(D)\} \in O(n)$. $\text{PREV}(D)$ on the other hand, can bundle each of the sets of size \sqrt{n} separately and sell it to the interested buyer, getting a total revenue of $\sqrt{n} \cdot \Omega(\sqrt{n} \log \sqrt{n}) = \Omega(n \log n)$. \square

6 One Buyer with Correlated Values

In this section, we study the relationship between $\text{SREV}(D)$, $\text{Max}\{\text{SREV}(D), \text{BREV}(D)\}$, and $\text{PREV}(D)$ for a single buyer with correlated values. The prior work of [17, 7] already shows that there is no hope of obtaining a finite bound between any of these quantities and $\text{REV}(D)$ because they are all deterministic, even when there are only two items. But it is still important to understand the relationship between these mechanisms of varying complexity even if their revenue cannot compare to that of the optimal mechanism. We show in Theorem 6 that for any distribution D for a single buyer, possibly even correlated, $\text{SREV}(D)$ is a $O(\log n)$ approximation to $\text{BREV}(D)$, and thus also to $\text{Max}\{\text{SREV}(D), \text{BREV}(D)\}$ and $\text{PREV}(D)$.¹⁰ We then show in Proposition 10 that this bound is tight, $\text{Max}\{\text{SREV}(D), \text{BREV}(D)\} \leq \text{PREV}(D)/\Omega(\log n)$. In other words, $\text{SREV}(D)$ provides a logarithmic approximation to $\text{PREV}(D)$, but taking $\max\{\text{SREV}(D), \text{BREV}(D)\}$ can't guarantee anything better.

We start by showing that $\text{SREV}(D)$ is a $O(\log n)$ approximation to $\text{BREV}(D)$. The proof of the theorem appears in Appendix F.

Theorem 6. *For any n -dimensional value distribution D for a single buyer (possibly correlated across items), $\text{BREV}(D) \leq 5 \ln(n) \text{SREV}(D)$. Therefore, $\text{PREV}(D) \leq 5 \ln(n) \text{SREV}(D)$ as well.*

Finally, we show that there is a setting with one buyer (and correlated item values) for which $\max\{\text{SREV}(D), \text{BREV}(D)\}$ provides poor approximation not only to $\text{REV}(D)$ but even to $\text{PREV}(D)$. The proof of the proposition appears in Appendix F.

Proposition 10. *There exists a (correlated) distribution D of the valuation of a single buyer over n items for which $\max\{\text{SREV}(D), \text{BREV}(D)\} \leq \text{PREV}(D)/\Omega(\log n)$.*

7 Acknowledgments

In an earlier version of this paper, we proved a factor of 7.5 in Theorem 2. This factor was later improved by Aviad Rubinstein to a factor of 6. We thank Aviad for allowing us to include this improvement in our paper.

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¹⁰As SREV approximate BREV for any set of items, it can do so for any part in the partition in PREV separately, and thus also approximate PREV .

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A Summary of Known Results

Table 1 and Table 2 presents the best results known for one additive buyer with item values sampled independently or arbitrarily, respectively. Additionally, Table 3 presents the best results known for many additive buyers in the independent setting. In each cell there is the known upper and lower bounds of the ratio between the corresponding column quantity and row quantity, and the source of the result. For example, in table 1, table entry that corresponds to the row marked by $\max\{\text{SREV}, \text{BREV}\}$ and column marked by REV there is the upper bound of 6 that we have presented in Theorem 2 for the ratio $\text{REV}/\max\{\text{SREV}, \text{BREV}\}$ which holds for every distribution D . Results from this paper that are not implied by other results appear in bold. Results that are implied from other results, point to the results that imply them.

Table 1: One buyer, independent item values

	$\max\{\text{SREV}, \text{BREV}\}$	REV
SREV	$O(\log n)$ [\rightarrow] $\Omega(\log n)$ [17]	$O(\log n)$ [22] $\Omega(\log n)$ [\leftarrow]
$\max\{\text{SREV}, \text{BREV}\}$	1 1	6 [Thm 2] $\Omega(1)$ [Folklore]

Table 2: One buyer, correlated item values

	$\max\{\text{SREV}, \text{BREV}\}$	PREV	REV
SREV	$O(\log n)$ [\rightarrow] $\Omega(\log n)$ [17]	$O(\log n)$ [Thm 6] $\Omega(\log n)$ [\downarrow]	0 ∞ [\downarrow]
$\max\{\text{SREV}, \text{BREV}\}$	1 1	$O(\log n)$ [\uparrow] $\Omega(\log n)$ [Prop 10]	0 ∞ [\downarrow]
PREV		1 1	0 ∞ [17]

B Omitted Proofs from Section 2

The proofs of Lemmas 1 and 2 require some technical lemmas from [17]. We include them below with proofs for completeness. In Lemma 8 below, D and D' are distributions over values for disjoint sets of items for the same consumers, and D and D' may be dependent. By

Table 3: Many buyers, independent item values

	$\max\{\text{SREV}, \text{BREV}\}$	PREV	REV
SREV	$O(\log n)$ [\rightarrow] $\Omega(\log n)$ [17]	$O(\log n)$ [\rightarrow] $\Omega(\log n)$ [\leftarrow]	$O(\log n)$ [Thm 3] $\Omega(\log n)$ [\leftarrow]
$\max\{\text{SREV}, \text{BREV}\}$	1 1	$O(\log n)$ [\uparrow] $\Omega(\log n)$ [Prop 9]	$O(\log n)$ [\uparrow] $\Omega(\log n)$ [\downarrow]
PREV		1 1	$O(\log n)$ [\uparrow] $\Omega(\log n)$ [Prop 8]

$\text{REV}(D, D')$ we mean the optimal revenue obtainable by selling to consumers whose values for items is sampled from the joint distribution according to D and D' .

Lemma 8. (“Marginal Mechanism” [17]) $\text{REV}(D, D') \leq \text{VAL}(D) + \mathbb{E}_{\vec{v} \leftarrow D}[\text{REV}(D'|\vec{v})]$.

Proof. We will establish a lower bound on $\mathbb{E}_{\vec{v} \leftarrow D} \text{REV}(D'|\vec{v})$ by constructing a truthful mechanism for selling items in the support of D' , based on one for those in the support of (D, D') . First, sample values \vec{v} from D for each consumer. Then announce that whenever a consumer would have received an item in the support of D , he will instead receive money equal to the (make-believe) value sampled from D . Note that, due to this announcement, each consumer now has a value for each item in the support of D corresponding to the sampled value profile. Then run the optimal truthful mechanism for selling items in the support of (D, D') . Conditioned on \vec{v} , this mechanism is truthful. Therefore, conditioned on \vec{v} , the revenue of this mechanism is upper bounded by $\text{REV}(D'|\vec{v})$. Taking an expectation over all $\vec{v} \leftarrow D$, we see that the total expected revenue obtained by this procedure is upper bounded by $\mathbb{E}_{\vec{v} \leftarrow D}[\text{REV}(D'|\vec{v})]$. Also taking an expectation over all $\vec{v} \leftarrow D$, we see that the expected revenue obtained is exactly $\text{REV}(D, D')$ minus the expected amount of money given away via the reduction, which upper bounded by $\text{VAL}(D)$. These two observations together prove the lemma. \square

Proof of Lemma 1: This is an immediate corollary of Lemma 8. as D and D' are independent, $\text{REV}(D'|\vec{v}) = \text{REV}(D')$, for all \vec{v} . \square

Lemma 9. (“Sub-Domain Stitching” [17]) Let S_1, \dots, S_k form a partition of \mathbb{R}^{nm} and let $s_i = \Pr[\vec{v} \in S_i | \vec{v} \leftarrow D]$. Then $\sum_i s_i \text{REV}(D|\vec{v} \in S_i) \geq \text{REV}(D)$.

Proof. Let M be the optimal mechanism for D , and $\text{REV}_M(D)$ denote the revenue of M when consumers are sampled from D . Then we have $\text{REV}_M(D) = \sum_i s_i \text{REV}_M(D|\vec{v} \in S_i)$. We also clearly have $\text{REV}(D) = \text{REV}_M(D)$, and $\text{REV}_M(D|\vec{v} \in S_i) \leq \text{REV}(D|\vec{v} \in S_i)$ for all i , proving the lemma. \square

In the lemma below, again think of D and D' as independent distributions for the same consumers over disjoint sets of items.

Lemma 10. (“Marginal Mechanism on Sub-Domain” [17]) S be any subset of \mathbb{R}^{nm} , and $s = \Pr[(\vec{v}, \vec{v}') \in S | (\vec{v}, \vec{v}') \leftarrow D \times D']$. Then $s \text{REV}(D \times D' | (\vec{v}, \vec{v}') \in S) \leq s \text{VAL}(D | (\vec{v}, \vec{v}') \in S) + \text{REV}(D')$.

Proof. Let $\mathbb{1}_E$ denote the indicator variable for event E (that is 1 when E occurs or 0 otherwise). Then when we write $\mathbb{1}_E D$, we mean the distribution that first samples $\vec{v} \leftarrow D$, and outputs \vec{v} if the event E occurs, and 0 otherwise. In particular, when we write $\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D \times D'$, we mean to first sample $(\vec{v}, \vec{v}') \leftarrow D \times D'$, and then output (\vec{v}, \vec{v}') if $(\vec{v}, \vec{v}') \in S$, or $\vec{0}$ otherwise. We'll also abuse notation slightly and write $\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D$ to mean that one should sample $(\vec{v}, \vec{v}') \leftarrow D \times D'$, and output only \vec{v} if $(\vec{v}, \vec{v}') \in S$ (or $\vec{0}$ otherwise). Then it's clear that $s\text{REV}(D \times D' | (\vec{v}, \vec{v}') \in S) = \text{REV}(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D \times D')$, which can be rewritten as $\text{REV}(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D, \mathbb{1}_{(\vec{v}, \vec{v}') \in S} D')$. Applying Lemma 8, we get:

$$\text{REV}(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D, \mathbb{1}_{(\vec{v}, \vec{v}') \in S} D') \leq \text{VAL}(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D) + \mathbb{E}_{\vec{v} \leftarrow \mathbb{1}_{(\vec{v}, \vec{v}') \in S} D} [\text{REV}(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D' | \vec{v})].$$

Now, define $S_{\vec{v}} = \{\vec{v}' | (\vec{v}, \vec{v}') \in S\}$. Then clearly, the distribution $(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D' | \vec{v})$ is exactly the same as $\mathbb{1}_{\vec{v}' \in S_{\vec{v}}} D'$, because D and D' are independent. Finally, it's clear that $\text{REV}(\mathbb{1}_{\vec{v}' \in S_{\vec{v}}} D') \leq \text{REV}(D')$, as any mechanism with no positive transfers makes at least as much revenue on D' as $\mathbb{1}_{\vec{v}' \in S_{\vec{v}}} D'$. So in conclusion, $\text{REV}(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D' | \vec{v}) \leq \text{REV}(D')$ for all \vec{v} , and therefore we have $\mathbb{E}_{\vec{v} \leftarrow \mathbb{1}_{(\vec{v}, \vec{v}') \in S} D} [\text{REV}(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D' | \vec{v})] \leq \text{REV}(D')$ as well.

The final step is simply observing that $\text{VAL}(\mathbb{1}_{(\vec{v}, \vec{v}') \in S} D) = s\text{VAL}(D | (\vec{v}, \vec{v}') \in S)$. \square

Proof of Lemma 2: We first prove the lemma in the case of $m = 1$ by induction on n .

Base Case, $n = 1$: It is clear that the lemma holds when $m = n = 1$ as $s\text{REV}(D) = \text{REV}(D)$ for all one-dimensional D .

Inductive Hypothesis: When $m = 1$, and $n = k$, $k s\text{REV}(D) \geq \text{REV}(D)$.

Inductive Step: Consider now the case where $n = k + 1$. Partition the support of D into two parts: S_1 where $v_1 \geq v_i$ for all i , and S_2 the remainder. Let also s_1 denote the probability that a buyer's valuation sampled from D lies in S_1 , and s_2 the probability that it lies in S_2 (so $s_1 + s_2 = 1$). Finally, let $D^{(i)}$ denote the distribution D conditioned on $\vec{v} \in S_i$.

An immediate corollary of Lemma 9 is that $\text{REV}(D) \leq s_1 \text{REV}(D^{(1)}) + s_2 \text{REV}(D^{(2)})$, so now our goal is to bound $s_1 \text{REV}(D^{(1)})$ and $s_2 \text{REV}(D^{(2)})$.

By Lemma 10, $s_1 \text{REV}(D^{(1)}) \leq s_1 \text{VAL}(D_{-1}^{(1)}) + \text{REV}(D_1)$. Furthermore, we claim that $s_1 \text{VAL}(D_{-1}^{(1)}) \leq k \text{REV}(D_1)$. To see this, observe that one truthful mechanism for selling just item 1 first samples $\vec{v}_{-1} \leftarrow D_{-1}$, and then sets a price of $\max_{i \geq 2} \{v_i\}$. Conditioned on $\vec{v} \in S_1$, the item will always sell (by the definition of S_1), and will generate revenue $\max_{i \geq 2} \{v_i\} \geq \frac{1}{k} \sum_{i \geq 2} v_i$. So the item sells with probability s_1 , and makes expected revenue at least $\frac{1}{k} \text{VAL}(D_{-1}^{(1)})$ when this occurs. So we have just shown that $\text{REV}(D_1) \geq \frac{s_1}{k} \text{VAL}(D_{-1}^{(1)})$, as claimed. Taken together with the observation at the beginning of this paragraph, we have now shown that $s_1 \text{REV}(D^{(1)}) \leq (k + 1) \text{REV}(D_1)$.

Now we just need to bound $s_2 \text{REV}(D^{(2)})$. Again by Lemma 10, $s_2 \text{REV}(D^{(2)}) \leq s_2 \text{VAL}(D_1^{(2)}) + \text{REV}(D_{-1})$. By the induction hypothesis, $\text{REV}(D_{-1}) \leq k s\text{REV}(D_{-1})$, as it is a product distribution over k items. So we just need to bound $s_2 \text{VAL}(D_1^{(2)})$. We claim that $s_2 \text{VAL}(D_1^{(2)}) \leq s\text{REV}(D_{-1})$. One way to sell items 2 through $k + 1$ separately is to first sample a value $v_1 \leftarrow D_1$, and set v_1 as the price for each item. Conditioned on $\vec{v} \in S_2$, at least one item will sell (by the definition of S_2), and so the revenue will be at least v_1 . So with probability s_2 at least one item sells, and the expected revenue is at least $\text{VAL}(D_1^{(2)})$ when this occurs. So we have just shown that $s\text{REV}(D_{-1}) \geq s_2 \text{VAL}(D_1^{(2)})$, as claimed. Taken together with the observation at the beginning of this paragraph, we have now shown that $s_2 \text{REV}(D^{(2)}) \leq (k + 1) s\text{REV}(D_{-1})$.

Taken together, we have now shown that $\text{REV}(D) \leq (k + 1) s\text{REV}(D_{-1}) + (k + 1) \text{REV}(D_1) = (k + 1) s\text{REV}(D)$, completing the inductive step.

At this point, we have now shown that for $m = 1$ and all $n \geq 1$, the statement of the lemma holds. We conclude by proving the $m > 1$ case. To extend to $m > 1$ buyers, observe that any truthful m -buyer mechanism M induces m truthful single-buyer mechanisms M_1, \dots, M_m such

that $\text{REV}(M) = \sum_j \text{REV}(M_j)$ (i.e. for each M_j , just sample $m - 1$ make-believe buyers and have them play M). As each single-buyer mechanism has revenue no more than $n\text{SREV}(D)$, the mechanism M cannot have revenue more than $nm\text{SREV}(D)$. \square

C Omitted Proofs from Section 3

Proof of Lemma 3: One could sell item i using a second price auction with reserve $t_i r_i$ to guarantee revenue at least $p_i t_i r_i$. If $p_i > 1/t_i$, then this contradicts the fact that the optimal obtainable revenue is r_i . \square

Proof of Lemma 4: The distribution D_i^C is stochastically dominated by D_i . Because only one item is for sale, this implies that the optimal revenue is also less.

It is possible to obtain revenue $p_i \text{REV}(D_i^T)$ when selling to consumers from D_i . Simply use whatever mechanism is used to obtain revenue $\text{REV}(D_i^T)$. With probability p_i , the consumers will be sampled from D_i^T and yield this much revenue. Therefore, we must have $p_i \text{REV}(D_i^T) \leq r_i$. \square

Proof of Lemma 5: This is a direct application of Lemma 9. Applied here, observe that the supports of D_A form a partition of the support of D when taken over all A . \square

D Omitted Proofs from Section 4

Proof of Lemma 7: $\text{var}(F) \leq E_{X \sim F}[X^2]$. As the optimal revenue of F is at most c , we know that $\Pr_{X \sim F}[X \geq x] \leq c/x$ for all x . So

$$\begin{aligned} E_{X \sim F}[X^2] &= \int_0^{t^2 c^2} \Pr_{X \sim F}[X^2 \geq x] dx \\ &\leq \int_0^{c^2} dx + \int_{c^2}^{t^2 c^2} c/\sqrt{x} dx \\ &= c^2 + 2c\sqrt{x} \Big|_{c^2}^{t^2 c^2} \\ &= c^2 + 2tc^2 - 2c^2 = (2t - 1)c^2. \end{aligned}$$

\square

E Omitted Proofs from Section 5

Proof of Proposition 5: The following proof is similar to that of Proposition 1, with two differences. First, we start with the bound $\text{REV}(D_A^T) \leq a|A|\text{SREV}(D_A^T)$ (instead of $\text{REV}(D_A^T) \leq |A|\text{SREV}(D_A^T)$). Second, we have to make use of the fact that when there is only one item, $\text{SREV}(D_A^T) = \text{REV}(D_A^T)$ and use this tighter bound whenever $|A| = 1$. We continue now with the proof.

By hypothesis and Lemma 4, for A with $|A| > 1$ it holds that $\text{REV}(D_A^T) \leq a|A|\text{SREV}(D_A^T) \leq \sum_{i \in A} a|A|r_i/p_i$. Combining with $\text{SREV}(D_A^T) = \text{REV}(D_A^T)$ for the case that $|A| = 1$, we can rewrite

$$\sum_A p_A \text{REV}(D_A^T) \leq \sum_i \left(r_i + \sum_{j=2}^n a_j \sum_{A \ni i, |A|=j} p_A r_i/p_i \right)$$

Observe that $p_A = (\prod_{i \in A} p_i)(\prod_{i \notin A} (1 - p_i)) \leq \prod_{i \in A} p_i$ and thus $p_A r_i/p_i \leq \prod_{k \in A - \{i\}} p_k r_i$. We then have that

$$\sum_{A \ni i, |A|=j} p_A r_i/p_i \leq r_i \sum_{A \ni i, |A|=j} \prod_{k \in A - \{i\}} p_k.$$

Furthermore, by Lemma 3, we have that each $p_i \leq 1/(c \cdot a \cdot n)$, so we have

$$\begin{aligned}
\sum_{j=2}^n \sum_{A \ni i, |A|=j} a_j \prod_{k \in A - \{i\}} p_k &\leq \sum_{j=2}^n a_j \binom{n-1}{j-1} / (c \cdot a \cdot n)^{j-1} \\
&\leq \sum_{j=2}^n \frac{j(n-1)^{j-1}}{(j-1)! c^{j-1} a^{j-2} n^{j-1}} \\
&\leq \sum_{j=2}^n \frac{2(j-1)}{(j-1)! c^{j-1} a^{j-2}} \\
&\leq \sum_{j=2}^n \frac{2}{(j-2)! c^{j-1} a^{j-2}} \\
&\leq \frac{2e^{1/ca}}{c}
\end{aligned}$$

The last inequality makes use of the fact that $\sum_{j=0}^{\infty} \frac{1}{j! c^j a^j}$ is the Taylor expansion for $e^{x/ca}$ evaluated at $x = 1$. Adding back the $j = 1$ term that we handled outside the sum (making use of the fact that SREV = REV on single-item distributions) and summing over all i of r_i times the above inequality yields the proposition. \square

Proof of Proposition 6: Note that $\text{VAL}(D_\emptyset^C) = \sum_i \text{VAL}(D_i^C) \leq \sum_i \int_0^{c n r_i} \text{Pr}[v_i^* > x] dx$. The last inequality would be equality if we replaced v_i^* with a random variable that is the maximum value in a sample drawn from D_i^C , but since v_i^* stochastically dominates such a random variable, we get an inequality instead. As the optimal revenue of D_i is r_i , this means that $\text{Pr}[v_i^* > x] \leq \min\{1, r_i/x\}$. So we have

$$\begin{aligned}
\text{VAL}(D_i^C) &\leq \int_0^{r_i} dx + \int_{r_i}^{c n r_i} r_i/x dx \\
&= r_i + r_i(\ln(c \cdot a \cdot n \cdot r_i) - \ln(r_i)) \\
&= r_i(1 + \ln n + \ln c + \ln a)
\end{aligned}$$

Summing this bound over all i yields the proposition. \square

Proof of Proposition 7: Recall from the proof of Proposition 1 that we may rewrite (using Corollary 2):

$$\sum_A p_A \text{REV}(D_A^T) \leq \sum_i r_i \sum_{A \ni i} 4|A|p_A = 4 \sum_i r_i \sum_{A \ni i} |A|p_A.$$

Again, the value $\sum_{A \ni i} |A|p_A$ is exactly the expected number of items in the tail, conditioned on i being in the tail. Therefore, $\sum_{A \ni i} |A|p_A \leq 1 + \sum_{j \neq i} p_j \leq 1 + 1/4$ by Lemma 3. Therefore, $\sum_A p_A \text{REV}(D_A^T) \leq 5 \sum_i r_i = 5\text{SREV}$. \square

E.1 Proof of Proposition 8

To complete the proof of Proposition 8 we only need to prove the following claim.

Lemma 11. *For the distribution D that is defined in Proposition 8 it holds that $\text{REV}(D) \in \Omega(n \log n)$ and $\text{PREV}(D) \in O(n)$.*

Proof. To see that $\text{REV}(D) \in \Omega(n \log n)$ consider the mechanism that sequentially visits the \sqrt{n} buyers, allowing each to pick any set of $\sqrt{n}/2$ items that are still available, and pay $c(\sqrt{n} \log n)$ for some $c > 0$ to be determined later. For each of the first $\sqrt{n}/4$ buyers, at least $7n/8$ items are remaining, and with very high probability the buyer has non-zero value for at least $\sqrt{n}/2$

of these items. This is because the expected number of such items is $7\sqrt{n}/8$, and the buyer has non-zero value for each item independently. Conditioned on having non-zero value for at least $\sqrt{n}/2$ items, Lemma 8 of [17] shows that with probability at least $1/2$, the buyer is willing to pay $c\sqrt{n} \log n$ for some constant c . Specifically, they show that the expected value of a consumer whose value for k items is sampled i.i.d. from Equal-Revenue distributions truncated at M is $\Theta(k \log M)$, and that the variance of the consumer's value is $O(M)$. Plugging in our choices of k and M , this means that the consumer's expected value for $\sqrt{n}/2$ items is $\Theta(\sqrt{n} \log n)$, and the variance is $O(n^{1/8})$. Meaning that with high probability, the consumer is willing to pay $\Omega(\sqrt{n} \log n)$. So the expected revenue from each of the first $\sqrt{n}/4$ buyers is at least $c\sqrt{n} \log n/2$, and therefore the total expected revenue from these buyers is $\Omega(n \log n)$.

To see that $\text{PREV}(D) \in O(n)$ consider the optimal partitioning mechanism, and denote the size of part k in the partition by s_k . Denote by $\text{BREV}(k)$ and $\text{SREV}(k)$ the revenue obtained by bundling together and selling separately within part k , respectively. Note that $\sum_k s_k = n$ and also that $\text{PREV}(D) = \sum_k \text{BREV}(k)$. For all k such that $s_k \leq n^{1/4}$, consider that the probability that a fixed consumer has non-zero value for 12 or more items is bounded above by taking a union bound over all $\binom{n^{1/4}}{12}$ subsets of size 12 for which the consumer might have non-zero value for every item, which occurs with probability $n^{-1/2}$. Taking another union bound over all \sqrt{n} consumers yields that the probability that any consumer has non-zero value for more than 12 items is at most $n^{-5/2}$. So even if we could achieve the maximum possible welfare ($= n^{1/4} * n^{1/8}$) whenever this occurred, it would contribute negligible expected revenue. When every consumer has non-zero value for 12 or fewer items in part k , we show that $24\text{SREV}(k) \geq \text{BREV}(k)$. A corollary of Theorem 5 in [13] is that the following auction obtains expected revenue $\text{BREV}(k)/2$: pick a single reserve price p , and let any buyer pay price p and take all items in part k . So for whatever price p guarantees $\text{BREV}(k)/2$, consider instead the mechanism that sets price $p/12$ for each item separately. Then clearly, whenever some consumer is willing to pay p for 12 or fewer items, he is also willing to pay $p/12$ for at least one item separately. So we get that $24\text{SREV}(k) \geq \text{BREV}(k)$. Also, a trivial upper bound on $\text{SREV}(k)$ is the revenue obtainable by selling goods separately without any supply constraint (i.e. the seller has infinitely many copies of each good instead of just one). The latter quantity is obtained by setting price 1 for each item to each buyer, yielding expected revenue exactly s_k . So we have now shown that $\text{BREV}(k) \leq 24s_k + o(1)$ whenever $s_k \leq n^{1/4}$.

On the other hand, maybe $s_k > n^{1/4}$. In this case, the additive Chernoff bound says that with probability at most e^{-2x^2} , the number of items for which a fixed buyer has non-zero value is at most $s_k/\sqrt{n} + \sqrt{s_k}x$.¹¹ Taking $x = \log n$, we get that except with negligible probability, every buyer has non-zero value for at most $s_k/\sqrt{n} + \sqrt{s_k} \log n$ items. As $s_k \geq n^{1/4}$, this is at most $2s_k \log(n)/n^{1/4}$. So even if some buyer had the maximum possible value for every item he valued above 0, his total value would be at most $2s_k \log(n)/n^{1/8} < s_k$. So $\text{BREV}(k) \leq s_k$.

So we have now shown that $\sum_k \text{BREV}(k) \leq 25n$, and therefore $\text{PREV}(D) \in O(n)$. \square

F Omitted Proofs from Section 6

F.1 Proof of Theorem 6

Theorem 6 follows directly from the three lemmas below. We first present two helpful definitions.

Definition 2. We say that an n -dimensional distribution D is a point-mass in sum distribution if when \vec{v} is sampled from D , $\sum_i v_i = p$ with probability 1.

Definition 3. We say that an n -dimensional distribution D is symmetric if all marginals D_i are the same.

¹¹This is obtained by plugging in $t = x\sqrt{s_k}$, and $E[\sum_i X_i] = s_k/\sqrt{n}$ to the bound $\Pr[\sum_{i=1}^{s_k} X_i \geq \mathbb{E}[\sum_{i=1}^{s_k} X_i] + t] \leq e^{-2t^2/s_k}$.

Lemma 12. *For any n -dimensional distribution D , there exists a point-mass in sum n -dimensional distribution D' such that $\text{BREV}(D')/\text{SREV}(D') \geq \text{BREV}(D)/\text{SREV}(D)$.*

Proof. Pick any instance D whose optimal grand bundle price is p , and the grand bundle sells at price p with probability q . We will transform D into a point-mass in sum distribution D' without decreasing the ratio $\text{BREV}(D)/\text{SREV}(D)$.

If \vec{v} denotes a sample from D , then observe that we may modify D to D'' such that $\text{BREV}(D'') \geq \text{BREV}(D)$ and $\text{SREV}(D'') \leq \text{SREV}(D)$ (and therefore $\text{BREV}(D'')/\text{SREV}(D'') \geq \text{BREV}(D)/\text{SREV}(D)$). Whenever $\sum_i v_i > p$, lower some values so that $\sum_i v_i = p$. Whenever $\sum_i v_i < p$, set all $v_i = 0$. It is clear that $\text{BREV}(D'') \geq \text{BREV}(D)$, as the consumer is still willing to pay p with probability q . It is also clear that $\text{SREV}(D'') \leq \text{SREV}(D)$ as we have only lowered the consumer's value for each item in a stochastically dominating way.

Next, define D' to be the distribution that is exactly D'' conditioned on $\sum_i v_i = p$. Then D'' samples from D' with probability q , and sets all values to 0 otherwise. It is also clear that $\text{SREV}(D'') = q\text{SREV}(D')$, because whatever price is set for each item sells with probability exactly q times the probability it sells when the consumer is drawn from D' (because the consumer will never pay anything for the item if instead all values are 0). Therefore, because $\text{BREV}(D'') = qp$, and $\text{BREV}(D') = p$, the two ratios are equal. That is: $\text{BREV}(D'')/\text{SREV}(D'') = \text{BREV}(D')/\text{SREV}(D')$. It is clear that D' is a point-mass in sum distribution. \square

Lemma 13. *For any n -dimensional distribution D , there exists a symmetric n -dimensional distribution D' such that $\text{BREV}(D')/\text{SREV}(D') \geq \text{BREV}(D)/\text{SREV}(D)$. If D was point-mass in sum, then D' is point-mass in sum as well.*

Proof. Define D' in the following way: sample \vec{v} from D , then randomly permute the components of \vec{v} to form \vec{v}' . It's clear that D' is symmetric. It's also clear that $\text{BREV}(D) = \text{BREV}(D')$. We just have to show that $\text{SREV}(D') \leq \text{SREV}(D)$. Let D_i denote the i^{th} marginal of D , D'_j denote the j^{th} marginal of D' , v_i denote a sample from D_i , and v'_j a sample from D'_j . Then D'_j samples from each D_i with probability $1/n$.

Now observe that $\text{SREV}(D') = \sum_j \max_p \{p \Pr[v'_j \geq p]\}$. As each D'_j samples each D_i with probability $1/n$, we get that $\Pr[v'_j \geq p] = \sum_i \Pr[v_i > p]/n$. this means that we can rewrite $\text{SREV}(D') = \sum_j \max_p \{p \sum_i \Pr[v_i > p]/n\} = \max_p \{p \sum_i \Pr[v_i > p]\}$. And observe also that $\text{SREV}(D) = \sum_i \max_p \{p \Pr[v_i \geq p]\}$. In other words, $\text{SREV}(D')$ is exactly $\text{SREV}(D)$ after swapping the order of the max and sum, which can only decrease $\text{SREV}(D')$. \square

Lemma 14. *Let D be any symmetric point-mass in sum distribution. Then $\text{BREV}(D) \leq 5 \ln(n) \text{SREV}(D)$.*

Proof. Without loss of generality, scale D down so that $\text{SREV}(D) = n$. We are essentially asking how large $\text{Welfare}(D)$ can possibly be subject to this, plus the symmetric point-mass in sum constraint. Denote $p = \text{Welfare}(D)$.

Note that each D_i is supported on $[0, p]$, and has expected revenue 1. So:

$$\text{Welfare}(D_i) = \int_0^p \Pr[v_i > x] dx \leq \int_0^1 dx + \int_1^p 1/x dx = 1 + \ln p$$

We now observe that we have two estimates of $\text{Welfare}(D)$. First, we know that $\text{Welfare}(D) = p$. And second, we know that $\text{Welfare}(D) = \sum_i \text{Welfare}(D_i) \leq n + n \ln p$. Putting these together, we get that p must satisfy:

$$p \leq n + n \ln p$$

For all $n \geq 2$, this implies that $p \leq 5n \ln n$ (as all $p > 5n \ln n$ violate the above inequality). \square

F.2 Proof of Proposition 10

Proof of Proposition 10: Assume that $n/(\log n)$ is an integer. Partition n into $\log n$ sets $S_1, \dots, S_{\log n}$, where $|S_k| = n/(\log n)$ for each k . The value distribution D is the following. For each set S_k independently, with probability $1 - n^{-2k}$ the value of every item is 0, and with probability n^{-2k} the value of each item is n^{2k} times a random value sampled i.i.d from a distribution F with CDF $F(x) = 1 - x^{-1}$ for $x \in [1, n]$ and $F(x) = 1$ for $x > n$ (an Equal-Revenue distribution with all mass above n moved to an atom at n). It is clear that $\text{SREV}(D) = n$, since the optimal expected revenue of each item is 1. To prove the claim we show that $\text{BREV}(D) \in O(n)$ and that $\text{PREV}(D) \in \Omega(n \log n)$.

To bound $\text{BREV}(D)$ from above, consider the revenue generated by a grand-bundle price of p . If $p \leq n$ then this revenue is at most n , so assume $p > n$. Choose $\ell \geq 0$ such that $p \in (n^{2\ell}, n^{2\ell+2}]$. First, observe that clearly the total value for $S_1 \sqcup \dots \sqcup S_{\ell-1}$ is at most $n^{2\ell}$, as there are only n items, and the value for each item is no more than $n^{2\ell-1}$. Next, observe that with high probability, no S_k for $k > \ell$ contributes any value. This is because by the union bound, the probability that any S_k , $k > \ell$ contributes non-zero value is at most $\sum_{k>\ell} n^{-2k} \leq 2n^{-2\ell-2}$. Next, we recall Lemma 8 of [17], which states that the bundling revenue obtained by selling x items to a buyer whose value for each item is sampled i.i.d. from the Equal-Revenue curve is $O(x \log x)$. Applied to S_ℓ , that means that for any $q > 0$, the probability that the buyer's value for S_ℓ exceeds q is $O(\min\{n^{-2\ell}, n/q\})$ (because $(n/\log n) \cdot \log(n/\log n) = O(n)$). Including back in the items in $S_1, \dots, S_{\ell-1}$, this means that the probability that the buyer's value for $S_1 \sqcup \dots \sqcup S_\ell$ exceeds $q + n^{2\ell}$ is also $O(\min\{n^{-2\ell}, n/q\})$. Finally, considering back S_k for $k > \ell$, we know that the probability that the buyer's value for the grand bundle exceeds $q + n^{2\ell}$ increases by at most $2n^{-2\ell-2}$, which is $O(1/p)$ for any $p \in (n^{2\ell}, n^{2\ell+2}]$. So for whatever price p we set, by taking $q = p - n^{2\ell}$ we get that the probability that the buyer's value for the grand bundle exceeds p is $O(n/p)$, and we conclude that $\text{BREV}(D) \in O(n)$.

To bound $\text{PREV}(D)$ from below, consider the mechanism that for each set S_k sets the optimal price for the bundle of all elements in that set. By Lemma 8 of [17] the optimal expected revenue from each set S_k is at least $(1/4)(n/(\log n)) \log(n/(\log n))$. Thus the total revenue of this mechanism, and hence $\text{PREV}(D)$, is at least $(1/4)n \log(n/(\log n)) = \Omega(n \log n)$. \square

G Computational Considerations

Our main result, Theorem 2, shows that $6 \cdot \max\{\text{SREV}(D), \text{BREV}(D)\} \geq \text{REV}(D)$ for a single buyer. This suggests a simple mechanism that obtains a constant approximation to the optimal revenue: estimate $\text{SREV}(D)$ and $\text{BREV}(D)$, then run whichever of the two mechanisms obtains higher revenue estimate. In this section we argue that a slight modification of this approach can be implemented in polynomial time, given appropriate access to the distribution D .

We will assume that we are given a sample access to the distributions $\{D_i\}_i$. Unfortunately, this kind of access cannot be sufficient to achieve any approximation, as a distribution might only return an exponentially large non-zero value with exponentially small probability. We thus make the additional weak assumption that for each D_i we also have access to a price p maximizing $p(1 - D_i(p))$.¹² Calculating $\text{SREV}(D)$ and implementing the mechanism that sells items separately is then trivial: one can directly optimize the revenue for each item, which yields $\text{SREV}(D)$ and the appropriate set of item prices.

Given that we can compute and implement $\text{SREV}(D)$, what we would like is to also estimate $\text{BREV}(D)$ and compute an approximately optimal price for the grand bundle. One approach would be to take samples from the distribution D , then optimize revenue for the observed

¹²For example, finding such a price is easy when each D_i is a discrete distribution given as a sequence of (value, probability) pairs.

empirical distribution. However, this strategy again suffers from the sample-complexity issues described above: the number of samples required to estimate $\text{BREV}(D)$ might, in principle, be quite large (again, certain items may have exponentially large value with exponentially small probability). To overcome this difficulty, we will not actually attempt to estimate $\text{BREV}(D)$; instead, we will argue that it suffices to estimate the expected revenue from selling the grand bundle at a *particular* price that we can compute explicitly. We will show that this revenue can be estimated sufficiently well with a small number of samples.

To see why such an approach is plausible, note that Proposition 3 and the proof of Proposition 4 together show that it isn't necessary to always estimate $\text{BREV}(D)$ accurately. All that is required for our approximation is that when the sum of values is concentrated around its expectation, and that expectation is large relative to $\text{SREV}(D)$ (case 2 in Proposition 4), then we must find a price for the grand bundle that extracts at least $1/4$ of that expectation as revenue. We will show that this can be achieved with a small number of samples.

More formally, consider the distribution D_\emptyset^C as used in Proposition 4. Proposition 3 plus the proof of Proposition 4 together establish that if $6 \cdot \text{SREV}(D) < \text{REV}(D)$, then it must be that

$$\Pr \left[\sum_i v_i \leq \frac{2}{5} \cdot \text{VAL}(D_\emptyset^C) \right] \leq \frac{25}{72}, \quad (1)$$

in which case we show that selling the grand bundle at a price of $\frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)$ yields at least a $1/6$ fraction of $\text{REV}(D)$. Since D_\emptyset^C is distribution D restricted to all values being at most $\text{SREV}(D)$ (which we can calculate), we can explicitly calculate the value of $\text{VAL}(D_\emptyset^C)$. We can then estimate $\Pr [\sum_i v_i \leq \frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)]$ by taking samples and testing explicitly. Standard concentration bounds imply that $\log n/\epsilon^2$ samples are sufficient to estimate this probability to within an additive error of ϵ , with high probability. Thus, with high probability, this number of samples is sufficient to determine whether inequality (1) (approximately) holds, and if so it yields an estimate (to within a multiplicative factor of $(1+\epsilon)$) of the revenue of selling the grand bundle at price $\frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)$. Comparing this revenue to $\text{SREV}(D)$, and implementing whichever scheme yields more revenue, obtains a $1/6(1+\epsilon)$ approximation to the optimal revenue.

To summarize, we have shown that the following algorithm obtains a $6(1+\epsilon)$ -approximation to the optimal revenue, for any $\epsilon > 0$:

1. Compute $\text{SREV}(D)$ and $p^* := \frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)$.
2. Take $\log n/\epsilon^2$ samples from D . Let q be the fraction of samples \vec{v} for which $[\sum_i v_i \geq p^*]$.
3. If $q < 47/72$, or if $q \cdot p^* < \text{SREV}(D)$, sell the items separately.
4. Otherwise, sell the grand bundle at price p^* .