

ANCILLA APPROXIMABLE QUANTUM STATE TRANSFORMATIONS

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ABSTRACT. We consider the transformations of quantum states obtainable by a process of the following sort. Combine the given input state with a specially prepared initial state of an auxiliary system. Apply a unitary transformation to the combined system. Measure the state of the auxiliary subsystem. If (and only if) it is in a specified final state, consider the process successful, and take the resulting state of the original (principal) system as the result of the process.

We review known information about exact realization of transformations by such a process. Then we present results about approximate realization of finite partial transformations. We consider primarily the issue of approximation to within a specified positive ε , but we also address the question of arbitrarily close approximation.

1. INTRODUCTION AND MAIN RESULTS

Consider an experiment involving the composition of two distinguishable quantum systems, a principal and an auxiliary one. Initially the auxiliary system is in a prepared initial state, and the principal system is in an arbitrary state $|\psi\rangle$. Apply a unitary operator U to the composite system, then measure the auxiliary system, and declare success if the auxiliary system is found to be in a particular (designated a priori) final state. In the case of success, let $\hat{U}|\psi\rangle$ be the resulting state of the principal system. The transformation \hat{U} is not necessarily unitary or even total.

Such an experiment is a recurring theme in recent quantum-computation literature; see [2, 3, 5, 6, 7, 9, 11, 16] for example. Typically one tries to maximize the probability that the measurement is successful and the state $\hat{U}|\psi\rangle$ is of some desired form, and one may or may not be able to use the resulting state of the principal system if the measurement is not successful. In particular, Childs and Wiebe use such an experiment to simulate convex linear combinations of unitary operators [6].

A number of natural questions arise including these:

- Which state transformations of the principal system can be exactly realized that way?
- What success probability can be guaranteed?
- How many ancillas are needed to achieve the desired results?

Much depends of course on the constraints imposed on the unitary operator U . In the simple case where no restrictions are placed on U , the answers to the three questions are known. We summarize them in the following Exact Realization Theorem. But first we need a few definitions.

Let $\mathcal{H}, \mathcal{H}^+$ be the Hilbert spaces for the principal and composite systems respectively. We presume that \mathcal{H}^+ is finite dimensional. If $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are the designated initial and final states of the auxiliary system and if the measurement is successful, then

$$\hat{U}|\psi\rangle = \pi_0 U(|\alpha_1\rangle \otimes |\psi\rangle)$$

where π_0 is the composition

$$\mathcal{H}^+ \rightarrow \{|\alpha_2\rangle\} \otimes \mathcal{H} \rightarrow \mathcal{H}$$

of a projection and an isomorphism, and the vector $\hat{U}|\psi\rangle$ is unnormalized. As in much of the literature, we usually ignore this distinction between a nonzero vector in a Hilbert space and the state represented by the vector, though we try to pay attention to the distinction in formal definitions and theorems. The following definition takes into account that nonzero collinear state vectors represent the same state. (Two vectors are collinear if one of them is a nonzero multiple of the other.)

Definition 1 (Exact Realization). A unitary operator U on \mathcal{H}^+ *exactly realizes* a partial transformation T of $\mathcal{H} - \{\vec{0}\}$ (into itself) if $\hat{U}|\psi\rangle$ is nonzero and collinear with $T|\psi\rangle$ for every $|\psi\rangle$ in the domain $\text{Dom}(T)$ of T .

The success probability $\text{SP}(U, |\psi\rangle)$ of U on a normalized \mathcal{H} vector $|\psi\rangle$ is $\|\hat{U}|\psi\rangle\|^2$. The *guaranteed success probability* of U is

$$\min\{\text{SP}(U, |\psi\rangle) : \|\psi\rangle\| = 1\}.$$

The use of \min (rather than \inf) is justified because the space of unitary operators is compact. Every linear operator on \mathcal{H} can be viewed as a partial transformation¹ of $\mathcal{H} - \{\vec{0}\}$.

¹By “partial”, we mean “not necessarily total”; so total transformations are a special case of partial ones.

Theorem 2 (Exact Realization). *Let L range over nonzero linear operators on \mathcal{H} , and let U range over unitary operators on \mathcal{H}^+ .*

- (1) *Every L is exactly realizable by some U , and one ancilla suffices for the purpose.*
- (2) *Let λ_{\min} and λ_{\max} be the minimal and maximal eigenvalues respectively of the positive operator $L^\dagger L$. If U exactly realizes L then the guaranteed success probability of U is at most $\frac{\lambda_{\min}}{\lambda_{\max}}$, and the upper bound is achieved by some unitary operators U .*

Even though Exact Realization Theorem is well-known to experts, we have found in the literature only a quick proof of Claim (1), namely the proof of Claim 6.2 in [1]. For the reader's convenience and to make this paper more self-contained, we give a detailed proof of the Exact Realization Theorem in §3. Specifically we need the following corollary of Theorem 2.

Corollary 3. *A partial transformation T of $\mathcal{H} - \{\vec{0}\}$ is exactly realizable if and only if there is a linear operator L on \mathcal{H} such that $L|\psi\rangle$ is nonzero and collinear with $T|\psi\rangle$ for every $|\psi\rangle \in \text{Dom}(T)$.*

Our main concern in this paper is with approximate realizability in the simple case of our experiment where no restrictions are placed on the unitary operator U on \mathcal{H}^+ . It will be convenient to identify nonzero collinear \mathcal{H} vectors and work in the resulting complex projective space \mathcal{P} where each point represents a unique state of the principal system, and each state is represented by a unique point in \mathcal{P} . We presume that $\mathcal{H} = \mathbb{C}^n$, so that \mathcal{P} is the complex projective space of (complex) dimension $n - 1$.

We show that, while almost every partial transformation of \mathcal{P} with domain of cardinality $\leq n + 1$ is approximately realizable, almost no partial transformation of \mathcal{P} with a larger domain is approximately realizable. To formulate this result precisely, we need a couple of definitions.

The point in \mathcal{P} given by a nonzero vector \vec{v} in \mathcal{H} will be denoted $Q\vec{v}$. Any linear transformation L of \mathcal{H} induces a partial transformation $Q\vec{v} \mapsto Q(L\vec{v})$ of \mathcal{P} , denoted QL , with $\text{Dom}(QL) = \{Q\vec{v} : L\vec{v} \neq \vec{0}\}$. The Q notation alludes to the fact that \mathcal{P} is a quotient of $\mathcal{H} - \{\vec{0}\}$. Corollary 3 justifies the following definition.

Definition 4 (Exactly realizable transformations of \mathcal{P}). A partial transformation τ of \mathcal{P} is *exactly realizable* if there is a linear transformation L of \mathcal{H} such that QL coincides with τ on $\text{Dom}(\tau)$.

The complex projective space \mathcal{P} is a Riemannian manifold endowed with the Fubini-Study metric, the only (up to a nonzero constant factor) Riemannian metric on \mathcal{P} invariant under (the transformations of \mathcal{P} induced by) unitary transformations of the overlying Hilbert space \mathcal{H} . The Fubini-Study metric induces the standard Fubini-Study distance measure

$$\text{FS}(Q\vec{u}, Q\vec{v}) = \arccos \frac{|\langle \vec{u} | \vec{v} \rangle|}{|\vec{u}| \cdot |\vec{v}|}.$$

It will be convenient to represent finite transformations of \mathcal{P} as point sequences. Fix a positive integer ℓ . A *suite* σ is a list $(p_1, \dots, p_\ell, p_{\ell+1}, \dots, p_{2\ell})$ of 2ℓ points where the first ℓ points p_1, \dots, p_ℓ are all distinct; it is a point in the direct product $\mathcal{P}^{2\ell}$ of the complex projective space \mathcal{P} . We think of it as specifying a transformation

$$p_i \mapsto p_{\ell+i} \quad \text{where } i = 1, \dots, \ell$$

and we say that it is exactly realizable if the transformation is. We carry over to suites the standard notation for domain and range of partial transformations; thus, we write $\text{Dom}(\sigma)$ for the first half, (p_1, \dots, p_ℓ) , of σ and $\text{Range}(\sigma)$ for the second half, $(p_{\ell+1}, \dots, p_{2\ell})$. Notice, though, that a suite contains more information than just the transformation that it specifies, because a suite also gives an ordering of its domain.

In what follows, ε ranges over positive real numbers.

Definition 5 (Approximately realizable suites).

- A suite $(q_1, \dots, q_{2\ell})$ ε -approximates a suite $(p_1, \dots, p_{2\ell})$ if every $\text{FS}(p_j, q_j) < \varepsilon$.
- A suite σ is ε -approximable if there is an exactly realizable suite τ that ε -approximates σ .
- A suite is *infinitely approximable* if it is ε -approximable for every $\varepsilon > 0$.

Theorem 6 (Approximate Realization). *In $\mathcal{P}^{2\ell}$, we have the following.*

- (1) *If $\ell \leq n + 1$ then the set of exactly realizable suites is a set of full measure.*
- (2) *If $\ell > n + 1$ then ε -approximable suites form an open set of volume $O(\varepsilon^{2(\ell-n-1)(n-1)})$.*

The proof of Claim (1) of Theorem 6 is elementary but the proof of Claim (2) involves the volume-of-the-tube theory pioneered by Hermann Weyl [8, 14] and Tarski's theorem about quantifier elimination in the first-order theory of algebraically closed fields [12].

Theorem 7 (Infinite Approximability). *If $\ell < 3$ then every suite is exactly realizable. Assume that $\ell \geq 3$ but the principal quantum system consists of just one qubit. A suite $(p_1, \dots, p_{2\ell})$ is infinitely approximable if and only if it is exactly realizable or else exactly $\ell - 1$ of the ℓ points $(p_{\ell+1}, \dots, p_{2\ell})$ are equal.*

The problem of characterization of infinitely approximable suites in the general case is open.

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2. PRESCRIBING INNER PRODUCTS

For the reader's convenience, we prove here some well-known facts about existence of vectors with prescribed inner products. We'll work over the complex field \mathbb{C} . Except when the contrary is explicitly stated, vector spaces of the form \mathbb{C}^d are assumed to be equipped with the standard (for physicists) inner product

$$\langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^d \bar{a}_i b_i.$$

Proposition 8. *Let Q be an $n \times n$ matrix of complex numbers. The following statements are equivalent.*

- (1) *For some positive integer d , there are n vectors $\vec{x}_i \in \mathbb{C}^d$ (where $i = 1, 2, \dots, n$) such that $\langle \vec{x}_i, \vec{x}_j \rangle = Q_{ij}$ for all i and j .*
- (2) *There are n vectors $\vec{x}_i \in \mathbb{C}^n$ (where $i = 1, 2, \dots, n$) such that $\langle \vec{x}_i, \vec{x}_j \rangle = Q_{ij}$ for all i and j .*
- (3) *Q is Hermitian, and $\sum_{i=1}^n \sum_{j=1}^n Q_{ij} \bar{z}_i z_j \geq 0$ for all $\vec{z} \in \mathbb{C}^n$.*
- (4) *Q is Hermitian, and all its eigenvalues are non-negative.*

Proof. We'll prove (1)→(3)→(2), and (3)→(4)→(3). Since (2) trivially implies (1), this will complete the proof.

(1)→(3): Given vectors \vec{x}_i as in (1), we have

$$Q_{ij} = \langle \vec{x}_i, \vec{x}_j \rangle = \overline{\langle \vec{x}_j, \vec{x}_i \rangle} = \overline{Q_{ji}},$$

so Q is Hermitian. We also have, for all $\vec{z} \in \mathbb{C}^n$, that

$$\sum_{i,j} Q_{ij} \bar{z}_i z_j = \sum_{i,j} \bar{z}_i \langle \vec{x}_i, \vec{x}_j \rangle z_j = \left\langle \sum_i z_i \vec{x}_i, \sum_j z_j \vec{x}_j \right\rangle \geq 0,$$

where the last inequality comes from the fact that the inner product of any vector with itself is non-negative.

(3)→(2): Assume (3) and consider an n -dimensional vector space V over \mathbb{C} with a basis $\{\vec{e}_1, \dots, \vec{e}_n\}$. (To avoid confusion, it is best *not* to identify V with \mathbb{C}^n at this stage; in particular, we do not want the standard inner product on V .) Define a sesquilinear form (i.e., linear in the second argument and conjugate-linear in the first) B on V by setting $B(\vec{e}_i, \vec{e}_j) = Q_{ij}$ and extending B to all vectors in V by sesquilinearity. Because Q is Hermitian, B is conjugate-symmetric, i.e., $B(\vec{u}, \vec{v}) = \overline{B(\vec{v}, \vec{u})}$.

Observe that the expression $\sum_{i,j} Q_{ij} \bar{z}_i z_j$, which we know to be non-negative by (3), is exactly $B(\sum_i z_i \vec{e}_i, \sum_i z_i \vec{e}_i)$.

Temporarily assume that this expression is not only non-negative but strictly positive for all $\vec{z} \neq \vec{0}$. Then B is an inner product on V . So we have an n -dimensional complex inner product space (namely V with inner product B) containing n vectors (namely the \vec{e}_i 's) whose inner products are given by Q_{ij} . But all n -dimensional inner-product spaces over \mathbb{C} are isomorphic, so the standard such space, \mathbb{C}^n with the standard inner product, must also contain such vectors. Thus, we have (2).

It remains to handle the case where $\sum_{i,j} Q_{ij} \bar{z}_i z_j$, though non-negative for all \vec{z} as required in (3), vanishes for some non-zero vectors \vec{z} . So B fails to be an inner product on V ; it satisfies all the requirements in the definition of inner products except that

$$K = \{\vec{u} \in V : B(\vec{u}, \vec{u}) = 0\}$$

is not merely $\{\vec{0}\}$.

We claim that $B(\vec{u}, \vec{v}) = 0$ whenever $\vec{u} \in K$, for all $\vec{v} \in V$. Indeed, for any such \vec{u} and \vec{v} and for any $\alpha \in \mathbb{C}$, we have

$$0 \leq B(\vec{v} + \alpha \vec{u}, \vec{v} + \alpha \vec{u}) = B(\vec{v}, \vec{v}) + 2\operatorname{Re}(\bar{\alpha} B(\vec{u}, \vec{v})).$$

If $B(\vec{u}, \vec{v})$ were not zero, then an appropriate choice of α would make $\operatorname{Re}(\bar{\alpha} B(\vec{u}, \vec{v}))$ so negative as to violate this inequality. This completes the proof of the claim that $B(\vec{u}, \vec{v}) = 0$ whenever $\vec{u} \in K$, for all $\vec{v} \in V$.

This claim has two consequences. First, it tells us that

$$K = \{\vec{u} \in V : (\forall \vec{v} \in V) B(\vec{u}, \vec{v}) = 0\}$$

and so K is a vector subspace of V . So we can form the quotient space V/K ; it is a complex vector space of dimension $< n$.

Second, we have, for arbitrary $\vec{u}, \vec{u}' \in K$ and arbitrary $\vec{v}, \vec{w} \in V$, that

$$B(\vec{v} + \vec{u}, \vec{w} + \vec{u}') = B(\vec{v}, \vec{w}).$$

This means that B determines a well-defined, conjugate-symmetric, sesquilinear form \hat{B} on V/K . That is, if we write $[\vec{v}]$ for the coset in

V/K that contains the vector \vec{v} , then

$$\hat{B}([\vec{v}], [\vec{w}]) = B(\vec{v}, \vec{w})$$

is well-defined and satisfies all the requirements for an inner product except perhaps positivity. It satisfies $\hat{B}([\vec{v}], [\vec{w}]) \geq 0$ because of the analogous fact about B . But also, by dividing out K , we have eliminated the danger of equality here. That is, if $\hat{B}([\vec{v}], [\vec{v}]) = 0$, then $B(\vec{v}, \vec{v}) = 0$, which means $\vec{v} \in K$ and so $[\vec{v}] = [\vec{0}]$. So \hat{B} is an inner product on V/K .

Again, we have a complex inner product space (namely V/K with \hat{B}) containing n vectors (namely the $[\vec{e}_i]$'s) whose inner products are given by the entries of Q . The same therefore holds of any other complex inner product space of the same dimension, since all such spaces are isomorphic. Since V/K has dimension $< n$, we can find appropriate vectors in \mathbb{C}^n (with room to spare), verifying (2).

(3) \rightarrow (4): Since Q is Hermitian, all its eigenvalues are real. If one of them were negative, say $\lambda < 0$ with eigenvector $\vec{z} \neq \vec{0}$, then

$$\sum_{i=1}^n \sum_{j=1}^n Q_{ij} \bar{z}_i z_j = \sum_{i=1}^n \bar{z}_i (Q\vec{z})_i = \lambda \sum_{i=1}^n \bar{z}_i z_i < 0,$$

contradicting the assumption (3).

(4) \rightarrow (3): Since Q is Hermitian, there is a unitary matrix U such that UQU^\dagger is a diagonal matrix D , whose diagonal entries are the eigenvalues of Q , known to be non-negative by (4). So we have $Q = U^\dagger D U$. For any $\vec{z} \in \mathbb{C}^n$, view \vec{z} as a column vector and observe that

$$\sum_{i,j} Q_{ij} \bar{z}_i z_j = \vec{z}^\dagger Q \vec{z} = \vec{z}^\dagger U^\dagger D U \vec{z} = \vec{w}^\dagger D \vec{w},$$

where we've introduced the abbreviation \vec{w} for $U\vec{z}$. Since D is diagonal, we have

$$\vec{w}^\dagger D \vec{w} = \sum_i D_{ii} \bar{w}_i w_i,$$

in which every summand is non-negative. This completes the verification of (3) and thus the proof of the proposition. \square

3. EXACT REALIZATION THEOREM

We use same name for a linear operator and its matrix when the vector basis is clear from the context. Let L range over nonzero linear operators on the Hilbert space $\mathcal{H} = \mathbb{C}^n$ for the principal system. L is *weakly contracting* if $\|L\vec{v}\| \leq \|\vec{v}\|$ for every vector $\vec{v} \in \mathcal{H}$. Further, let

λ_{\min} and λ_{\max} be the minimal and maximal eigenvalues of the positive operator $L^\dagger L$.

3.1. Literal realization. We start by introducing a particular simple version of exact realization. Recall that, according to §1, every unitary operator U on the Hilbert space \mathcal{H}^+ for the composite system gives rise to a linear operator \hat{U} on \mathcal{H} .

Definition 9. A unitary operator U on \mathcal{H}^+ *literally realizes* L if $L = \hat{U}$.

Proposition 10. *The following statements are equivalent.*

- (1) L is literally realizable.
- (2) L is literally realizable with one ancilla.
- (3) All eigenvalues of $L^\dagger L$ are ≤ 1 .
- (4) L is weakly contracting.

Proof. Clearly (2) \rightarrow (1). Taking into account that the Hermitian operator $L^\dagger L$ is diagonalizable, we see that (3) is equivalent to

$$(3') \text{ All eigenvalues of } I - L^\dagger L \text{ are } \geq 0.$$

In the rest of the proof, we establish (1) \rightarrow (3') \rightarrow (2) as well as (3) \leftrightarrow (4)

Let k be the dimension of the Hilbert space for the auxiliary system. We work in some basis $|0\rangle, \dots, |kn - 1\rangle$ of \mathcal{H}^+ . To simplify the exposition, we presume (without loss of generality really) that the initial state $|\alpha_1\rangle$ and the final state $|\alpha_2\rangle$ of the auxiliary system coincide, and that the first n basic states $|0\rangle, \dots, |n - 1\rangle$ of the composite system are exactly the basic states where the auxiliary system is in state $|\alpha_1\rangle$. According to §1,

$$\hat{U}|\psi\rangle = \iota\pi U(|\alpha_1\rangle \otimes |\psi\rangle)$$

where π is the projection $\mathcal{H}^+ \rightarrow \{|\alpha_1\rangle\} \otimes \mathcal{H}$ and ι is the isomorphism $\{|\alpha_1\rangle\} \otimes \mathcal{H} \rightarrow \mathcal{H}$.

(1) \rightarrow (3') Assume $L = \hat{U}$. The matrix πU is obtained from matrix U by leaving the upper n rows intact and zeroing the other entries; the lower $kn - n$ rows of U play little role in our proof. Further, only the upper n entries of the vector $|\alpha_1\rangle \otimes |\psi\rangle$ may be nonzero, and so the right $kn - n$ columns of matrix U play little role in our proof. If M is the upper left $n \times n$ minor of U then $M\vec{v} = L\vec{v}$ for all vectors $\vec{v} \in \mathcal{H}$. Thus matrix L is the upper left minor of matrix U .

Let X be the lower left $(kn - n) \times n$ submatrix of U (the submatrix right under the minor L), and let $|L_1\rangle, \dots, |L_n\rangle$ and $|X_1\rangle, \dots, |X_n\rangle$ be

the columns of L and X respectively. Since U is unitary, we have

$$\langle X_i | X_j \rangle = \begin{cases} -\langle L_i | L_j \rangle & \text{if } i \neq j, \\ 1 - \langle L_i | L_j \rangle & \text{if } i = j, \end{cases}$$

so that the matrix $X^\dagger X = I - L^\dagger L$. By the implication (2)→(4) of Proposition 8, with $I - L^\dagger L$ playing the role of Q , all eigenvalues of $I - L^\dagger L$ are non-negative.

(3') → (2) Assume (3'). By the implication (4) → (1) of Proposition 8, with $I - L^\dagger L$ playing the role of Q , there exist n -dimensional vectors $|X_1\rangle, \dots, |X_n\rangle$ such that the inner products $\langle X_i | X_j \rangle$ form the matrix $I - L^\dagger L$.

Now we are ready to construct the desired matrix U . Put L in the upper left corner of the matrix. Right under L put the $n \times n$ matrix with columns $|X_1\rangle, \dots, |X_n\rangle$. This gives us the first n columns of U which form an orthonormal basis B for an n -dimensional subspace of \mathcal{H}^+ . Extend the list B with the standard basis $|0\rangle, \dots, |2n-1\rangle$ for \mathcal{H}^+ and then apply the Gram-Schmidt algorithm to the resulting list in order to obtain an orthonormal basis for \mathcal{H}^+ extending B . This basis provides the columns of the desired matrix U . Thus claims (1),(2),(3) are equivalent. To finish the proof, it suffices to establish that (3) \iff (4).

(3) → (4) Assume (3). Let vectors $|e_i\rangle$ form an orthonormal basis of eigenvectors of $L^\dagger L$ with eigenvalues $\lambda_1, \dots, \lambda_n$ respectively. Then $|\psi\rangle$ is a linear combination $\sum_i \alpha_i |e_i\rangle$. We have

$$\begin{aligned} \|L|\psi\rangle\|^2 &= (L|\psi\rangle)^\dagger (L|\psi\rangle) = \langle \psi | L^\dagger L | \psi \rangle \\ &= \sum_{i,j} \bar{\alpha}_j \alpha_i \langle e_j | L^\dagger L | e_i \rangle = \sum_{i,j} \bar{\alpha}_j \alpha_i \lambda_i \langle e_j | e_i \rangle \\ &= \sum_i |\alpha_i|^2 \lambda_i \leq \sum_i |\alpha_i|^2 = \|\psi\|^2. \end{aligned}$$

Next assume (4) and let $|\psi\rangle$ be an eigenvector of $L^\dagger L$ with some eigenvalue λ . Then

$$\begin{aligned} \lambda \|\psi\|^2 &= \langle \psi | \lambda | \psi \rangle = \langle \psi | L^\dagger L | \psi \rangle \\ &= (L|\psi\rangle)^\dagger (L|\psi\rangle) = \|L|\psi\rangle\|^2 \leq \|\psi\|^2, \end{aligned}$$

so $\lambda \leq 1$. □

Corollary 11. [6] *Any convex combination of unitary operators is literally realizable.*

Proof. Suppose L is a convex combination of some unitary operators U_i . Then, for any vector $|\psi\rangle$, $L|\psi\rangle$ is a convex combination (with the same coefficients) of the vectors $U_i|\psi\rangle$, each of which has the same length as $|\psi\rangle$. Since balls in Hilbert space are convex, it follows that $L|\psi\rangle$ has at most the same length as $|\psi\rangle$. So L is weakly contracting. By Proposition 10, L is literally realizable. \square

Childs and Wiebe prove more in [6]. In particular, the U that literally realizes a convex combination of two U_i 's can be computed by a circuit consisting of (a) unitary operators that act only on the ancilla and (b) the controlled U_i gates.

3.2. Literal realization vs. exact realization.

Proposition 12. *A unitary operator U on \mathcal{H}^+ exactly realizes a nonzero linear operator L if and only if it literally realizes some nonzero multiple cL of L .*

Proof. The if part of the proposition is obvious: if $cL = \hat{U}$ then $L\vec{v}$ and $\hat{U}\vec{v}$ are collinear for every $\vec{v} \in \mathcal{H}$. To prove the only-if part, we need an auxiliary result from linear algebra.

Lemma 13. *Let D, R be finite-dimensional complex vector spaces, and let A, B be linear transformations from D to R such that $A\vec{v}$ and $B\vec{v}$ are collinear for every $\vec{v} \in D$. Then A, B are collinear, that is $A = cB$ for some nonzero c .*

Proof of Lemma 13. First we treat the case where B is one-to-one. Let d be the dimension of D . If $d = 1$, the lemma is obvious, so we may assume that $d \geq 2$. Let vectors $\vec{v}_1, \dots, \vec{v}_d$ in D form a basis in D . Since B is one-to-one, the vectors $B\vec{v}_i$ are linearly independent. By the collinearity premise, there are nonzero complex numbers c_i such that $A\vec{v}_i = c_i B\vec{v}_i$. It suffices to show that all the numbers c_i are equal.

For any $i < j$, let $\vec{u} = \vec{v}_i + \vec{v}_j$. By the collinearity premise, $A\vec{u} = cB\vec{u}$ for some c . We have

$$\begin{aligned} A\vec{u} &= cB\vec{u} = cB\vec{v}_i + cB\vec{v}_j, \\ A\vec{u} &= A(\vec{v}_i + \vec{v}_j) = c_i B\vec{v}_i + c_j B\vec{v}_j, \end{aligned}$$

so that $cB\vec{v}_i + cB\vec{v}_j = c_i B\vec{v}_i + c_j B\vec{v}_j$. But vectors $B\vec{v}_i, B\vec{v}_j$ are independent. Then $c = c_i$ and $c = c_j$ and therefore $c_i = c_j$.

Second we treat the case where B is not one-to-one. Without loss of generality we may suppose that B is nonzero. Clearly, $A\vec{u} = \vec{0}$ whenever $B\vec{u} = \vec{0}$. That is, A vanishes on the kernel K of B . So we can regard both A and B as being defined on the quotient D/K , and

of course B is one-to-one on D/K , so that the preceding discussion applies.

Therefore there is a nonzero complex number c such that $A = cB$ on D/K . We check that $A = cB$ on D . Pick any nonzero vector $\vec{v} \in D$. Obviously $A\vec{v} = cB\vec{v}$ if $\vec{v} \in K$. Suppose that $\vec{v} \notin K$. By the collinearity premise, $A\vec{v} = c'B\vec{v}$ for some nonzero complex number c' . This equality results in a similar equality $A[\vec{v}] = c'B[\vec{v}]$ in the quotient D/K where we also have $A[\vec{v}] = cB[\vec{v}]$. Since vector $B[\vec{v}]$ is not zero, it follows that $c' = c$. \square

Now we are ready to prove the only-if part of the proposition. Assume that U exactly realizes L , so that $\hat{U}\vec{v}$ is nonzero and collinear with $L\vec{v}$ whenever $L\vec{v} \neq \vec{0}$. If the implication

$$L\vec{v} = \vec{0} \rightarrow \hat{U}\vec{v} = \vec{0}.$$

holds then, by Lemma 13, some nonzero multiple cL of L coincides with \hat{U} and therefore is literally realizable. Thus, it suffices to prove the implication.

Suppose $L\vec{v} = \vec{0}$. Since L is nonzero, there is a vector \vec{w} orthogonal to the kernel of L , so $L\vec{w}$ and $L(\vec{v} + \vec{w})$ are equal and nonzero. Hence $\hat{U}\vec{w}$ and $\hat{U}(\vec{v} + \vec{w})$ are nonzero and collinear, so $\hat{U}\vec{v} = b\hat{U}\vec{w}$ and therefore $\hat{U}(\vec{v} - b\vec{w}) = \vec{0}$ for some b . But then $L(\vec{v} - b\vec{w}) = \vec{0}$, so that $b = 0$ and $\hat{U}\vec{v} = \vec{0}$. \square

3.3. Guaranteed success probability.

Proposition 14. *If a unitary operator U literally realizes L then the guaranteed success probability of U is the least eigenvalue λ_{\min} of $L^\dagger L$.*

Proof. Let $|\psi\rangle$ range over the unit sphere of \mathbb{C}^n . Recall from §1 that the guaranteed success probability of U is $\min_{|\psi\rangle} \text{SP}(U, |\psi\rangle)$ where $\text{SP}(U, |\psi\rangle) = \|\hat{U}|\psi\rangle\|^2$, and assume that U literally realizes L . Then

$$\text{SP}(U, |\psi\rangle) = \|L|\psi\rangle\|^2.$$

There exist eigenvectors $|e_1\rangle, \dots, |e_n\rangle$ of $L^\dagger L$, with eigenvalues $\lambda_{\max} = \lambda_1 \geq \dots \geq \lambda_n = \lambda_{\min}$ respectively, that form an orthonormal basis for \mathcal{H} . An arbitrary unit vector $|\psi\rangle$ in \mathcal{H} is a linear combination

$\sum_{i=1}^n \alpha_i |e_i\rangle$. We have

$$\begin{aligned} \text{SP}(U, |\psi\rangle) &= \|L|\psi\rangle\|^2 = (L|\psi\rangle)^\dagger(L|\psi\rangle) = \langle\psi|L^\dagger L|\psi\rangle \\ &= \sum_{i,j} \bar{\alpha}_j \alpha_i \langle e_j | L^\dagger L | e_i \rangle = \sum_{i,j} \bar{\alpha}_j \alpha_i \lambda_i \langle e_j | e_i \rangle \\ &= \sum_i |\alpha_i|^2 \lambda_i \geq \lambda_{\min} \sum_i |\alpha_i|^2 = \lambda_{\min}. \end{aligned}$$

In particular $\text{SP}(U, |e_n\rangle) = \lambda_n = \lambda_{\min}$. \square

Corollary 15. *Suppose that U literally realizes L . Then L is invertible if and only if the guaranteed success probability of U is positive.*

3.4. Proof of Exact Realization Theorem.

Proof of Claim (1) of Theorem 2. If some nonzero multiple cL of the given linear operator L on \mathcal{H} is literally realizable then, by Proposition 10, cL is literally realizable by some unitary operator U with just one ancilla. But then U exactly realizes L , and one ancilla suffices. Thus it suffices to find a complex number $c \neq 0$ such that cL is literally realizable.

If $\lambda_{\max} \leq 1$ set $c = 1$; otherwise set $c = 1/\sqrt{\lambda_{\max}}$. In either case, by Proposition 10, cL is literally realizable. \square

Proof of Claim (2) of Theorem 2. Assume that U exactly realizes L . By Proposition 12, U literally realizes some nonzero multiple $M = cL$ of L . Let μ_{\min} and μ_{\max} be the minimal and maximal eigenvalues of $M^\dagger M$ respectively. Taking into account that M is nonzero and invoking Proposition 10, we have

$$0 < \mu_{\max} = |c|^2 \lambda_{\max} \leq 1.$$

By Proposition 14, the guaranteed success probability of U is

$$\mu_{\min} = |c|^2 \lambda_{\min} \leq \lambda_{\min} / \lambda_{\max}.$$

There is a real $d \geq 1$ such that $|cd|^2 \lambda_{\max} = 1$. Redefine M from cL to cdL . The unitary dU literally realizes M and therefore exactly realizes L . We have

$$0 < \mu_{\max} = |cd|^2 \lambda_{\max} = 1,$$

and the guaranteed success probability of dU is

$$\mu_{\min} = |cd|^2 \lambda_{\min} = \lambda_{\min} / \lambda_{\max}.$$

\square

4. APPROXIMATE REALIZATION THEOREM

In this section and in the rest of the paper, we use notation and definitions from §1. In particular, every nonzero vector $\vec{v} = (a_1, \dots, a_n)$ in \mathcal{H} represents a point $Q\vec{v}$ in \mathcal{P} . The complex numbers a_1, \dots, a_n are the homogeneous coordinates of $Q\vec{v}$; at least one of the homogeneous coordinates is nonzero. Further, any linear transformation L of \mathcal{H} induces a partial transformation QL of \mathcal{P} . If L is invertible then QL is total. Such total transformations QL are known as projective linear.

4.1. Projective linear transformations. As usual, nonzero vectors $\vec{v}_1, \dots, \vec{v}_m$ in \mathbb{C}^n are said to be *in general position* if, for any $k \leq m$, any k of the m vectors are linearly independent. Points $Q\vec{v}_1, \dots, Q\vec{v}_k$ are *in general position* if the vectors $\vec{v}_1, \dots, \vec{v}_k$ are so.

Lemma 16. *If points p_1, \dots, p_{n+1} are in general position and points q_1, \dots, q_{n+1} are in general position then there is a unique projective linear transformation g such that every $g(p_i) = q_i$.*

Proof. Let \vec{e}_{n+1} be the sum $\vec{e}_1 + \dots + \vec{e}_n$ of the basic vectors in \mathcal{H} . It is easy to check that vectors $\vec{e}_1, \dots, \vec{e}_{n+1}$ are in general position. It suffices to prove that for any vectors $\vec{v}_1, \dots, \vec{v}_{n+1}$ in general position there exists a unique, up to a constant factor, invertible linear operator L on \mathcal{H} such that every $L\vec{e}_i$ is collinear with \vec{v}_i .

First we prove the uniqueness. Suppose that L is a linear operator such that every $L\vec{e}_i$ is collinear with \vec{v}_i , and so there are nonzero complex numbers z_i such that

$$L\vec{e}_i = z_i\vec{v}_i \quad \text{for } i = 1, \dots, n + 1.$$

In the basis $\vec{e}_1, \dots, \vec{e}_n$, the column vector \vec{e}_i with $i \leq n$ has 1 at row i and zeroes everywhere else, so that the i^{th} column of the desired L is $z_i\vec{v}_i$. Since $\vec{e}_{n+1} = \vec{e}_1 + \dots + \vec{e}_n$, we have

$$z_{n+1}\vec{v}_{n+1} = \sum_{i=1}^n z_i\vec{v}_i.$$

Since vectors $\vec{v}_1, \dots, \vec{v}_n$ are independent, $\vec{v}_{n+1} = a_1\vec{e}_1 + \dots + a_n\vec{e}_n$ for some complex numbers a_1, \dots, a_n , so that $z_1 = a_1z_{n+1}, \dots, z_n = a_nz_{n+1}$. Since vectors $\vec{v}_0, \dots, \vec{v}_n$ are in general position, the coefficients a_1, \dots, a_n are nonzero. Let L_0 be the invertible matrix with columns $a_1\vec{v}_1, \dots, a_n\vec{v}_n$. Then $L = z_{n+1}L_0$.

Second we prove the existence. To this end, check that every $L_0\vec{e}_1 = a_1\vec{v}_1, \dots, L_0\vec{e}_n = a_n\vec{v}_n$ and $L_0\vec{e}_{n+1} = \vec{v}_{n+1}$. \square

Recall that suites are points of $\mathcal{P}^{2\ell}$ where the first ℓ coordinates are distinct and that a suite $\tau = (p_1, \dots, p_{2\ell})$ specifies the transformation $\tau(p_j) = p_{\ell+j}$ with domain $\{p_1, \dots, p_\ell\}$.

Definition 17 (PL manifold). The *projective linear manifold* PL consists of the suites specifying partial transformation of \mathcal{P} that can be extended to projective linear transformations of \mathcal{P} .

We say that, over \mathcal{H} , a sequence M_1, M_2, \dots of linear operators converges to a linear operator L if, for every vector \vec{v} , the sequence $M_i \vec{v}$ converges to $L\vec{v}$.

Lemma 18. *Over \mathcal{H} , for every linear operator L on \mathcal{H} there is a sequence M_1, M_2, \dots of invertible linear operators that converges to L .*

Proof. Without loss of generality, L is positive. Indeed, by the Polar Decomposition Theorem, $L = UL'$ for some unitary U and positive L' . If invertible linear operators M'_i converge to L' then $UM'_i \rightarrow UL'$.

Fix an orthonormal basis for \mathcal{H} composed of eigenvectors of L . In that basis, L is represented by a diagonal matrix. The (matrix for the) desired M_i is obtained from L by replacing every zero on the diagonal with $1/i$. \square

Proposition 19 (PL approximants suffice). *For every ε -approximable suite σ there is a PL suite that ε -approximates σ .*

Proof. Given an ε -approximable suite σ , first choose an exactly realizable suite τ that ε -approximates σ . Let δ be the maximum of the Fubini-Study distances between corresponding components of σ and τ . So $\delta < \varepsilon$. By Corollary 3, we have a linear operator L that realizes τ . By Lemma 18, we can find invertible linear operators M arbitrarily close to L . Taking M close enough to L , we can ensure, thanks to the continuity of the quotient map $Q : \mathcal{H} \rightarrow \mathcal{P}$, that QM maps each point in $\text{Dom}(\tau)$ to within $\varepsilon - \delta$ of the corresponding point in $\text{Range}(\tau)$. Then, letting τ' be the suite with the same domain half as τ but the range half given by applying QM to the domain, we get that τ' is within $\varepsilon - \delta$ of τ and therefore within ε of σ . \square

4.2. The PL manifold. The complex projective space \mathcal{P} has dimension $n - 1$. So $\dim(\mathcal{P}^{2\ell}) = 2\ell(n - 1)$.

Lemma 20 (Dimension of PL).

- (1) *If $\ell \leq n + 1$ then PL is an open set of full measure in $\mathcal{P}^{2\ell}$, and so $\dim(PL) = 2\ell(n - 1)$.*
- (2) *If $\ell > n + 1$ then $\dim(PL) \leq (n - 1)(\ell + n + 1)$.*

Proof of Lemma. Claim (1) follows from Lemma 16. We prove Claim (2).

A PL suite $\tau = (p_1, \dots, p_{2\ell})$ is determined by p_1, \dots, p_ℓ and an invertible linear operator L on $\mathcal{H} = \mathbb{C}^n$ such that $QL(p_j) = p_{\ell+j}$ for $j \leq \ell$. So PL is the range of a (smooth, in fact rational in local coordinates) map from $\mathcal{P}^\ell \times \mathcal{L}$ where \mathcal{L} is the space of linear operators on \mathbb{C}^n modulo scalar multiples. Thus

$$\dim(\text{PL}) \leq \dim(\mathcal{P}^\ell) + \dim(\mathcal{L}) = \ell(n-1) + (n^2-1) = (n-1)(\ell+n+1).$$

This completes the proof of the lemma. \square

We remark that the upper bound in Claim (2) of the lemma is, in all nontrivial cases (i.e., $n > 1$), strictly below the dimension of $\mathcal{P}^{2\ell}$.

It will be convenient to work in affine spaces rather than projective ones. To this end, cover the complex projective space \mathcal{P} by its n standard coordinate patches A_1, \dots, A_n . Here A_i consists of those points in \mathcal{P} whose i^{th} homogeneous coordinate is not zero; multiplying by a scalar, we can arrange that the i^{th} homogeneous coordinate is 1, and then we can use the remaining $n-1$ homogeneous coordinates as affine coordinates on A_i . The space $\mathcal{P}^{2\ell}$ is covered by $n^{2\ell}$ coordinate patches A that are the cartesian products of the coordinate patches in the 2ℓ factors.

Proposition 21 (Variety for PL). *In any coordinate patch A of $\mathcal{P}^{2\ell}$, there exist a full-measure open set G and an algebraic variety V such that $PL \subseteq V$ and $G \cap V \subseteq PL$.*

Here an algebraic variety is the set of solutions of a system of polynomial equations over the field \mathbb{C} of complex numbers. Notice that the union of two varieties is a variety. For example,

$$\begin{aligned} (f_1 = 0 \wedge f_2 = 0) \vee (g_1 = 0 \wedge g_2 = 0) &\iff \\ (f_1g_1 = 0) \wedge (f_1g_2 = 0) \wedge (f_2g_1 = 0) \wedge (f_2g_2 = 0). \end{aligned}$$

Proof. It is clear from the definition of PL that the intersection $A \cap \text{PL}$ is definable, in terms of the affine coordinates of A , in the first-order language of the field \mathbb{C} . By Tarski's theorem [12], the first-order definition of $A \cap \text{PL}$ can be rewritten in quantifier-free form. We can also arrange that the quantifier-free definition is in disjunctive normal form, and we can assume that each disjunct is satisfied by some points, because any other disjuncts could simply be omitted from the disjunctive normal form.

Any disjunct δ is a conjunction of some polynomial equations and some inequations. (“Inequation” here means \neq , whereas “inequality”

traditionally means $<, >, \leq, \geq$.) We can further arrange that there is at most one inequation among the conjuncts, because $(f \neq 0) \wedge (g \neq 0)$ is equivalent to $fg \neq 0$, and that there is at least one inequation, because if there is none then we can adjoin $1 \neq 0$. So δ has the form

$$f_1 = f_2 = \cdots = f_k = 0 \wedge g \neq 0,$$

where all f_j as well as g are polynomials. Let $E_\delta = \text{Set}(g = 0)$. Here $\text{Set}(g = 0)$ is the solution set for $g = 0$ in A ; we will use similar notation for other formulas as well. Further, let E be the union, over all disjuncts δ , of the sets E_δ . The desired full-measure open set is $G = A - E$.

Since there are only finitely many disjuncts, it suffices to prove that, for any disjunct δ , there is an algebraic variety V in A such that $\text{Set}(\delta) \subseteq V$ and $G \cap V \subseteq \text{PL}$ is a set of measure 0.

To obtain the desired V , we simply remove the inequation from δ , so that $V = \text{Set}(f_1 = \cdots = f_k = 0)$. This looks simplistic but it works. Obviously V is an algebraic variety and V includes $\text{Set}(\delta)$. Further,

$$\begin{aligned} V &= [V \cap \text{Set}(g \neq 0)] \cup [V \cap \text{Set}(g = 0)] \\ &= \text{Set}(\delta) \cup [V \cap \text{Set}(g = 0)] \\ &\subseteq \text{PL} \cup E_\delta \end{aligned}$$

and therefore $G \cap V \subseteq \text{PL}$. □

4.3. Tubes. The purpose of this subsection is to provide some information on tubes that we'll need in the proof of Approximate Realization Theorem.

Consider a one-dimensional curve L in a three-dimensional cube. Given a small $\varepsilon > 0$ and a point $p \in L$, form a disc of radius ε , within the ambient cube, centered at p and orthogonal to L at p . (For simplicity, we ignore the possibility that the disc bulges beyond the cube. More pedantically, we should be talking about the portion of the disc within the cube.) As p traverses the curve L , the disc traverses a three-dimensional tube of radius ε around L .

Similarly the ε -approximable points of $\mathcal{P}^{2\ell}$ form a tube, Tube_ε around the set of exactly approximable suites. By Proposition 19, it is also a tube around the PL manifold.

If the curve L is nice, one may expect that the volume of the three-dimensional tube of radius ε is about $\pi\varepsilon^2$ times the length of L . One can estimate the volume of Tube_ε in a similar way. But the curve L can be so curly that its length is infinite; the classical example is the curve $\sin(1/x)$ where $0 < x < 1$ which has a singularity at $x = 0$. The curve may even fill in the whole cube. Of course, the PL manifold does

not look bad, and there is a well developed theory of tubes around a submanifold of a given manifold [8]. The PL manifold has singularities, and it is not obvious at all how to apply the general theory of tubes. Fortunately, by Proposition 21, the PL manifold is, up to a set of measure zero, a finite union of algebraic varieties. That helps.

Theorem 22 (Wongkew’s theorem [17]). *Let V be an algebraic variety of codimension k in the m -dimensional Euclidean space given by polynomials of degree $\leq d$. Then there exist constants c_k, \dots, c_m which depend only on m , so that for any ball B of radius R and any positive ε , the volume of the ε -tube around $B \cap V$ is bounded by $\sum_{j=k}^m (c_j d^j R^{m-j}) \varepsilon^j$.*

Corollary 23. *Let V be an algebraic variety of complex codimension k in a finite-dimensional Hilbert space, and let ε range over positive reals. For any bounded open set X (or its closure), the volume of the tube of radius ε around $X \cap V$ is $O(\varepsilon^{2k})$.*

Proof. We get $O(\varepsilon^{2k})$ by taking into account only the leading term in the sum in Wongkew’s theorem. The exponent is doubled because we work in a Hilbert space and k is the complex dimension whereas Wongkew’s theorem refers to real dimensions. Finally, balls can be replaced with bounded open sets because the closure of such a set is compact and therefore is covered by finitely many balls. \square

4.4. Proof of Approximate Realization Theorem.

Proof of Claim (1) of Theorem 6. Assume $\ell \leq n + 1$ and let G be the set of general-position suites, so that a suite $(p_1, \dots, p_{2\ell}) \in G$ if and only if the 2ℓ points $p_1, \dots, p_{2\ell}$ are in general position. Clearly G is an open set of measure 1. By Proposition 19, $G \subseteq \text{PL}$, so the PL manifold is of full measure. But every PL suite is exactly realizable. So the set of exactly realizable suites is of full measure. \square

Proof of Claim (2) of Theorem 6. Assume $\ell \geq n + 2$, and define the *unit cube* of a Hilbert space \mathbb{C}^m to comprise the points in \mathbb{C}^m whose coordinates all have absolute values in the real interval $[0, 1]$.

Recall the coordinate patches of \mathcal{P} and $\mathcal{P}^{2\ell}$ that we used in the proof of Proposition 21. If you identify a patch A_i of \mathcal{P} with a copy of \mathbb{C}^{n-1} , then it makes sense to speak about a unit cube C_i in A_i . The cubes C_1, \dots, C_n cover \mathcal{P} . Indeed, if $p \in \mathcal{P}$ and the i^{th} homogeneous coordinate of p is, in absolute value, a largest homogeneous coordinate of p then p belongs to the cube C_i .

Now, every coordinate patch A of $\mathcal{P}^{2\ell}$ is a cartesian product of coordinate patches in the 2ℓ factors of $\mathcal{P}^{2\ell}$. View every factor patch as

a copy of \mathbb{C}^{n-1} . Then A is a copy of $\mathbb{C}^{2\ell(n-1)}$. The unit cube C in A is the cartesian product of the unit cubes in the 2ℓ factor patches. It follows that $\mathcal{P}^{2\ell}$ is covered by the unit cubes in its coordinate patches.

Since the number of such cubes is finite, it suffices to prove the following for every coordinate patch A of $\mathcal{P}^{2\ell}$: The ε -approximable suites form an open set of volume $O(\varepsilon^{2(\ell-n-1)(n-1)})$ in the unit cube C of A . By Proposition 19, it suffices to prove that, in C , the volume of the ε -tube around $C \cap \text{PL}$ is $O(\varepsilon^{2(\ell-n-1)(n-1)})$.

To this end, let G and V_1, \dots, V_k be as in Proposition 21. Then $C \cap G$ is an open set of measure 1 in the cube C and $(C \cap G) \cap \text{PL} = (C \cap G) \cap (V_1 \cup \dots \cup V_k)$.

Since the set $C - G$ is of measure 0, it suffices to prove that, in $C \cap G$, the volume of the ε -tube around $(C \cap G) \cap \text{PL}$ is $O(\varepsilon^{2(\ell-n-1)(n-1)})$. To this end, it suffices to prove that, in $C \cap G$, the volume of the ε -tube around every $C \cap V_j$ is $O(\varepsilon^{2(r-1)(n-1)})$, but this follows directly from Corollary 23. \square

5. INFINITE APPROXIMABILITY

Recall that a suite is infinitely approximable if it is ε -approximable for every positive ε . By Proposition 19, the infinitely approximable suites form the closure of PL.

We start with a couple of general remarks and then give a complete characterization of infinitely approximable suites in the case where the principal quantum system consists of a single qubit. The problem of characterization of infinitely approximable suites in the general case is open.

5.1. General considerations. Recall that the Hilbert space \mathcal{H} for our principal quantum system is \mathbb{C}^n and the corresponding projective space is $\mathcal{P} = \mathbb{C}\mathbb{P}^{n-1}$. Suites are tuples in $\mathcal{P}^{2\ell}$ where the first ℓ points are distinct. A suite $\sigma = (p_1, \dots, p_{2\ell})$ is viewed as the finite transformation that sends the domain tuple $\text{Dom}(\sigma) = (p_1, \dots, p_\ell)$ to the range tuple $\text{Range}(\sigma) = (p_{\ell+1}, \dots, p_{2\ell})$. If σ is infinitely approximable then every vicinity of σ contains exactly realizable suites; but σ itself does not have to be exactly realizable.

Example 1 (An infinitely approximable suite that is not exactly realizable). Set $\ell = n + 1$. In this case, by Lemma 16, every suite in general position extends to a projective linear transformation and thus is exactly realizable. Since general-position suites form an open set of full measure, every suite is infinitely approximable. It remains to construct a suite $\sigma = (p_1, \dots, p_{2n+2})$ that is not exactly realizable.

Construction. Given an orthonormal basis $\vec{e}_1, \dots, \vec{e}_n$ for \mathcal{H} , let $\vec{e}_{n+1} = \sum_{i=1}^n \vec{e}_i$. We saw, in the proof of Lemma 16, that the vectors $\vec{e}_1, \dots, \vec{e}_{n+1}$ are in general position. Set

$$p_1 = Q\vec{e}_1, \dots, p_n = Q\vec{e}_n, p_{n+1} = Q\vec{e}_{n+1}$$

$$\sigma(p_1) = \dots = \sigma(p_n) = p_1, \sigma(p_{n+1}) = p_2.$$

By reduction to absurdity, assume that σ extends to the transformation QL for some linear operator L on \mathcal{H} . Then the vectors $L\vec{e}_1, \dots, L\vec{e}_n$ are collinear with \vec{e}_1 , and so the range of L is the one-dimensional subspace spanned by \vec{e}_1 . Accordingly the range of QL consists of a single point p_1 while the range of σ contains p_2 as well. \square

Lemma 24. *For each suite σ of length $2n + 2$, whose domain half $Dom(\sigma)$ and range half $Range(\sigma)$ are each in general position, let f_σ be the unique projective linear transformation that maps $Dom(\sigma)$ to $Range(\sigma)$. Then $f_\sigma(p)$ is a continuous function of σ (in the space of suites) and p (in \mathcal{P}).*

Proof. Because continuity is a local property, we may assume that the relevant points, namely the $2n + 2$ components of σ and the point p , are each confined to lie in one of the n coordinate patches that cover \mathcal{P} . (Of course, different components might be in different patches.) Fixing these patches, we can fix a normalization for the homogeneous coordinates of the relevant points. If a point is confined to the patch where the i^{th} homogeneous coordinate is non-zero, then we normalize its homogeneous coordinates so that the i^{th} coordinate is 1.

We now revisit the proof of Lemma 16, paying attention to continuity issues.

To begin, consider suites of the form $(E, Range(\sigma))$, where the range is that of a variable σ , as above, but the domain is fixed as the $(n + 1)$ -tuple $(Q\vec{e}_1, \dots, Q\vec{e}_{n+1})$ of points in \mathcal{P} corresponding to the n standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$ and their sum \vec{e}_{n+1} in \mathcal{H} . Because $Range(\sigma)$ is in general position, the proof of Lemma 16 produces an invertible matrix L corresponding to the projective linear transformation $f_{(E, Range(\sigma))}$, and now we need to look more closely at this L . (Recall that it is unique up to an overall nonzero scalar factor.) It can be obtained as follows. First form the matrix L' whose columns are the homogeneous coordinates (normalized as above) of the first n components of $Range(\sigma)$. The corresponding projective linear transformation transforms each $Q\vec{e}_i$ for $i = 1, 2, \dots, n$ correctly, namely to $Range(\sigma)_i$, but it might transform $Q\vec{e}_{n+1}$ incorrectly. To correct this one remaining component, without damaging the other n , we multiply the columns of L' by suitable nonzero scalars z_i . Any choice of z_i 's will preserve

the correctness of the values at $Q\vec{e}_i$ for $i = 1, 2, \dots, n$, but the z_i 's must be chosen carefully to ensure that the new matrix L sends e_{n+1} to $\text{Range}(\sigma)_{n+1}$. (It would suffice to send e_{n+1} to a vector collinear with $\text{Range}(\sigma)_{n+1}$, but this additional freedom is just the freedom, already noted above, to multiply L by an overall nonzero scalar factor.) The required condition on the z_i 's is a system of linear equations, whose coefficient matrix is L' . The fact that L' is invertible ensures not only that there is a unique solution for the z_i 's but also that this solution is a continuous function of $\text{Range}(\sigma)$. Indeed, by Cramer's rule, the solution is given by certain rational functions, namely ratios of determinants, of the entries of L' and the components of $\text{Range}(\sigma)_{n+1}$. Since the entries of L' are components of $\text{Range}(\sigma)_i$ for $i = 1, \dots, n$, and since the denominator of these rational expressions, the determinant of L' , is not zero, we have the claimed continuity of the z_i 's. It follows that L is a continuous function of $\text{Range}(\sigma)$.

Similarly, we can realize the finite transformation $(\text{Dom}(\sigma), E)$ by a matrix M whose entries are continuous functions of $\text{Dom}(\sigma)$. Indeed, the previous paragraph shows how to continuously realize $(E, \text{Dom}(\sigma))$. To realize $(\text{Dom}(\sigma), E)$, we need only take the inverse matrix. It will still be a continuous function of $\text{Dom}(\sigma)$, because matrix inversion is a continuous function, given by ratios of determinants.

Having realized both $(E, \text{Range}(\sigma))$ and $(\text{Dom}(\sigma), E)$ by matrices that depend continuously on σ , we need only multiply these matrices (and observe that multiplication is continuous) to realize σ .

Finally, $f_\sigma(p)$ can be obtained as the image in \mathcal{P} of the product of the matrix realizing σ and the column vector (normalized as above) representing p . It is therefore a continuous function of σ and p . \square

The *border* of PL consists of the infinitely approximable suites that do not belong to PL.

Claim 25. *If σ is a suite on the border of PL, then there cannot be $n + 1$ points in general position in $\text{Dom}(\sigma)$ such that the corresponding $n + 1$ points in $\text{Range}(\sigma)$ are also in general position.*

Proof. Suppose that $\sigma = (p_1, \dots, p_l, q_1, \dots, q_l)$ were a counterexample. To simplify the notation, permute the components, if necessary, so that (p_1, \dots, p_{n+1}) and (q_1, \dots, q_{n+1}) are general-position $(n + 1)$ -tuples. By Lemma 16, let f be the unique projective linear transformation that sends p_i to q_i for all i in the range $1 \leq i \leq n + 1$. We shall show that $f(p_j) = q_j$ also for $n + 1 < j \leq l$. This will complete the proof, because it means that f realizes σ and therefore σ belongs to PL, not to its border as assumed. For the rest of the proof, we fix some arbitrary

j in the relevant range, $n + 1 < j \leq l$, and our goal is to prove that $f(p_j) = q_j$.

Since σ is in the closure of PL, we can consider PL suites σ' arbitrarily close to σ . Temporarily consider a fixed σ' near σ . (Later, we shall let σ' vary and approach σ .) Let us write τ for the suite $(p_1, \dots, p_{n+1}, q_1, \dots, q_{n+1})$ of length $2(n+1)$; so f realizes τ . Similarly, let us write τ' for the suite consisting of the first $n+1$ points from the domain and from the range of σ' . Since σ' belongs to PL, it is realized by some projective linear transformation f' . Of course this f' also realizes τ' . Furthermore, f' sends the j^{th} component p'_j of σ' to the corresponding component q'_j in the range half of σ' (the $(l+j)^{\text{th}}$ component of σ').

Now let σ' vary, in PL, and approach the border suite σ . Then in particular, τ' approaches τ , p'_j approaches p_j , and q'_j approaches q_j . Applying Lemma 24 (with τ' in the role of the σ in the lemma), we find that $f'(p'_j)$ approaches $f(p_j)$. That is, q'_j approaches $f(p_j)$. But, since σ' approaches σ , we also know that q'_j approaches q_j . Therefore, $f(p_j) = q_j$, as required. \square

Now we turn to the single-qubit case where $\mathcal{H} = C^2$ and \mathcal{P} is the Riemann sphere $= \mathbb{C}\mathbb{P}^1$ that extends the field \mathbb{C} of complex numbers with an additional point ∞ .

In this case, Claim 25 simplifies somewhat, because a tuple is in general position if and only if all its components are distinct. Indeed, since $n = 2$ in this case, the definition of general position requires simply that any two of the components are images, in \mathcal{P} , of independent vectors in \mathcal{H} , which means that they are distinct points in \mathcal{P} .

Our definition of “suite” requires the components in the domain half to be distinct, so Claim 25 has the following consequence.

Corollary 26. *In the single-qubit case, every suite on the border of PL has at most two distinct points in its range half.*

A point of \mathcal{P} with homogeneous coordinates (a, b) can be conveniently represented as the ratio a/b where $a/b = \infty$ if $b = 0$.

The Riemann-sphere representation of one-qubit states is closely related to the Bloch-sphere [11], a representation of one-qubit states on the unit sphere \mathcal{S}^2 of the three dimensional Euclidean space \mathbb{E}^3 . Recall the standard stereographic projection of \mathcal{S}^2 — from the north pole onto the plane through the equator. Think of this plane as a copy of \mathbb{C} . Then the standard stereographic projection naturally extends to the stereographic projection of \mathcal{S}^2 onto the Riemann sphere by mapping the north pole onto ∞ . Let π be the inverse projection of the Riemann

sphere onto the Bloch sphere. It is easy to check that $\pi Q|\psi\rangle$ is the Bloch-sphere representation of the state given by the vector $|\psi\rangle$ in \mathcal{H} . In particular $\pi\infty$ is the north pole of \mathcal{S}^2 . The Fubini-Study distance between points z_1, z_2 on the Riemann sphere is one half of the geodesic distance between the points $\pi z_1, \pi z_2$ on Bloch sphere.

Lemma 27. *If L is a nonzero linear operator on \mathcal{H} given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the standard orthonormal basis of \mathcal{H} then QL is the transformation $z \mapsto \frac{az + b}{cz + d}$.*

Proof. Let $\vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2$ and $z = \alpha/\beta$. If $\beta \neq 0$ then we have

$$(QL)z = Q\left(L\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) = Q\left(L\begin{pmatrix} z \\ 1 \end{pmatrix}\right) = Q\left(\begin{pmatrix} az + b \\ cz + d \end{pmatrix}\right) = \frac{az + b}{cz + d}.$$

If $\beta = 0$ then $z = \infty$, and we have

$$(QL)\infty = Q\left(L\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = Q\left(\begin{pmatrix} a \\ c \end{pmatrix}\right) = \frac{a}{c} = \frac{a\infty + b}{c\infty + d}.$$

□

5.2. Cross-ratio. Projective linear transformations of the Riemann sphere are known as fractional linear transformations and have the form $\frac{c_1z + c_2}{c_3z + c_4}$ where c_1, \dots, c_4 are complex numbers with $c_1c_4 - c_2c_3 \neq 0$.

The cross-ratio of four distinct points a, b, c, d on the Riemann sphere is defined by

$$\chi(a, b, c, d) = \frac{a - c}{b - c} \cdot \frac{b - d}{a - d}.$$

If one of the four points is ∞ , the cross-ratio is defined by continuity; that amounts to just omitting those two of the four factors that involve ∞ . It is easy to check that the cross-ratio is invariant under fractional linear transformations.

Although defined for tetrads of distinct points (general position), the cross-ratio extends continuously to tetrads in which two of the four points are equal while the other two are distinct (configuration $2 + 1 + 1$), and also to tetrads in the configuration $2 + 2$, provided we allow ∞ as a value for the cross-ratio. By Corollary 26, configuration $2 + 1 + 1$ cannot occur in the range of a suite on the border of PL, but configuration $2 + 2$ is consistent with the corollary.

Lemma 28. *Let a, b, c, d be points in the Riemann sphere \mathcal{P} .*

- (1) *If the tetrad (a, b, c, d) is $2 + 2$ then $\chi(a, b, c, d) \in \{0, 1, \infty\}$.*
- (2) *If a, b, c, d are distinct then $\chi(a, b, c, d) \notin \{0, 1, \infty\}$.*

Proof. To prove claim (1), check that

- if $a = b \neq c = d$ then $\chi(a, b, c, d) = 1$,
- if $a = c \neq b = d$ then $\chi(a, b, c, d) = 0$,
- if $a = d \neq b = c$ then $\chi(a, b, c, d) = \infty$.

We prove claim (2) by reductio ad absurdum. Let a, b, c, d be arbitrary distinct points in \mathcal{P} .

First suppose that $\chi(a, b, c, d) \in \{0, \infty\}$. If all points a, b, c, d are complex numbers (not ∞) then clearly $\chi(a, b, c, d) \notin \{0, \infty\}$. If $a = \infty$ then $\chi(a, b, c, d) = \frac{b-d}{b-c} \notin \{0, \infty\}$. The cases $b = \infty$, $c = \infty$ and $d = \infty$ are similar.

Second suppose that $\chi(a, b, c, d) = 1$. This is equivalent to each of the following equations:

$$\begin{aligned} (a - c)(b - d) &= (b - c)(a - d) \\ ab - ad - bc + cd &= ab - bd - ac + cd \\ bd + ac - ad - bc &= 0 \\ (a - b)(c - d) &= 0. \end{aligned}$$

So either $a = b$ or $c = d$. □

Every suite σ of length 8 consists of a domain tetrad $\text{Dom}(\sigma)$, where all four points are distinct, and a range tetrad $\text{Range}(\sigma)$.

Lemma 29. *No suite σ of length 8 such that $\text{Range}(\sigma)$ is $2 + 2$ is a limit point of the PL manifold of suites of length 8.*

Proof. By reductio ad absurdum, suppose that σ is a suite of length 8 such that $\text{Range}(\sigma)$ is $2 + 2$ and σ is a limit point of the PL manifold of suites of length 8. Then there is a sequence τ_1, τ_2, \dots of PL suites of length 8 that converges to σ . In particular, the domain tetrads $\text{Dom}(\tau_k)$ of suites τ_k converge to $\text{Dom}(\sigma)$, and the range tetrads $\text{Range}(\tau_k)$ of suites τ_k converge to $\text{Range}(\sigma)$. By continuity, cross-ratios $\chi(\text{Dom}(\tau_k)) \rightarrow \chi(\text{Dom}(\sigma))$ and cross-ratios $\chi(\text{Range}(\tau_k)) \rightarrow \chi(\text{Range}(\sigma))$ as $k \rightarrow \infty$. Since fractional linear transformations preserve cross-ratios, every $\chi(\text{Dom}(\tau_k)) = \chi(\text{Range}(\tau_k))$, and so

$$\begin{aligned} \chi(\text{Dom}(\sigma)) &= \lim_{k \rightarrow \infty} \chi(\text{Dom}(\tau_k)) \\ &= \lim_{k \rightarrow \infty} \chi(\text{Range}(\tau_k)) \\ &= \chi(\text{Range}(\sigma)) \end{aligned}$$

which contradicts Lemma 28 □

In contrast to configurations $2 + 1 + 1$ and $2 + 2$, the cross-ratio does not extend continuously to tetrads in the configurations $3 + 1$ or 4 .

Indeed, any neighborhood of any tetrad in either of these configurations contains general-position tetrads with all possible cross-ratios.

To see this, it suffices to prove the claim for one tetrad of each of these two sorts, say $(0, 0, 0, \infty)$ and $(0, 0, 0, 0)$. (This sufficiency follows immediately from the facts that the group of fractional linear transformations acts transitively on each of these two sorts of tetrads (because it acts doubly transitively² on the Riemann sphere) and preserves cross-ratios.) Given any possible cross-ratio, we can find two tetrads (a, b, c, d) and (a', b', c', ∞) with that cross-ratio, where all of a, b, c, d, a', b', c' are complex numbers (not ∞). Now apply to these tetrads the fractional linear transformation $z \mapsto \varepsilon z$ for a very small, positive, real ε . The resulting tetrads have the same cross-ratio and are very close — arbitrarily close as $\varepsilon \rightarrow 0$ — to $(0, 0, 0, 0)$ and $(0, 0, 0, \infty)$, respectively, as claimed.

5.3. Proof of Infinite Approximability Theorem. We assume that $\ell \geq 3$ and the principal quantum system consists of just one qubit. Fix an orthonormal basis \vec{e}_1, \vec{e}_2 in $\mathcal{H} = \mathbb{C}^2$.

Lemma 30. *A suite σ is exactly realizable by means of a nonzero singular linear operator L on \mathcal{H} (so that σ extends to QL) if and only if all ℓ points in $\text{Range}(\sigma)$ are equal.*

Proof. If a nonzero singular linear operator L exactly realizes the given suite σ then $L\vec{e}_1$ and $L\vec{e}_2$ are collinear and QL is constant. Hence all points in $\text{Range}(\sigma)$ are equal.

If all points in $\text{Range}(\sigma)$ are equal, say to a point $Q\vec{v}$, then the desired L can be obtained by setting $L\vec{e}_1 = L\vec{e}_2 = \vec{v}$. \square

Proposition 31 (Border of PL). *A suite σ belongs to the border of the PL manifold if and only if the range part $\text{Range}(\sigma)$ satisfies one of the following two conditions.*

- (1) *All ℓ points in $\text{Range}(\sigma)$ are equal.*
- (2) *Exactly $\ell - 1$ of the ℓ points in $\text{Range}(\sigma)$ are equal.*

Proof. We first prove the only-if implication. Suppose that σ belongs to the border of PL. By Claim 25, the range tuple $\text{Range}(\sigma)$ contains at most two distinct points. If all points in $\text{Range}(\sigma)$ are equal, we are done. Suppose that $\text{Range}(\sigma)$ contains exactly two distinct points. Then the index set $\{\ell + 1, \dots, 2\ell\}$ splits into disjoint parts I and J such that the same point p occurs in all I positions and a different point q occurs in all J positions. Without loss of generality, I contains at least two indices. Suppose toward a contradiction that J contains at least

²In fact it acts triply transitively, but that's not relevant here.

two indices as well, so that $\ell \geq 4$. Then there is a suite σ_0 of length 8 embedded in the suite σ of length 2ℓ such that the range of σ_0 is of type $2+2$ and σ_0 is a limit point of the PL manifold of suites of length 8. This contradicts Lemma 29.

Next we prove the if implication. Notice that, in either of the two cases, σ does not belong to PL. Indeed, fractional linear transformations (and projective linear transformations in general) preserve equality and disequality, and so all points in the range part of a PL suite are distinct. It remains to prove that σ belongs to the closure of PL.

If all points in $\text{Range}(\sigma)$ are equal then, by the preceding lemma, σ is exactly realizable by means of a nonzero singular linear operator L . Now use Lemma 18.

Suppose that some $\ell - 1$ points in $\text{Range}(\sigma)$ are equal, say to a point p , but another point q also occurs in $\text{Range}(\sigma)$. It suffices to consider the case where $p = 0$ and $q = \infty$. Indeed, there is a fractional linear transformation f that moves p, q to $0, \infty$ respectively. If PL suites τ_k converge to $f\sigma$ then PL suites $f^{-1}\tau_k$ converge to σ .

Without loss of generality, ∞ occurs in the very last position in σ , so that σ has the form

$$(p_1, \dots, p_{\ell-1}, p_\ell, 0, \dots, 0, \infty).$$

There is a fractional linear transformation g that sends p_ℓ to ∞ . For each $k = 1, 2, \dots$, let g_k be the fractional linear transformation g/k , and let τ_k be the restriction of g_k to $\text{Dom}(\sigma)$. The sequence of PL suites τ_k converges to σ . \square

Theorem 7 follows from Proposition 31 and Lemma 30.

6. FINAL REMARKS

6.1. Mixed states. We have been working with pure states. One may consider a generalization of the results above to mixed states. Here we just point out that the scenario in the beginning of our story readily generalizes to mixed states and channel representation.

As is, the scenario is not a channel; since the unsuccessful measurement result is discarded, the trace is not preserved. But the scenario becomes a channel if the unsuccessful measurement result is not discarded and the measurement result is not looked at. The modified scenario corresponds to a composition of three channels, as follows. To simplify the exposition, we consider only the case of one principal qubit and one ancilla, and we presume that the designated initial and final states of the ancilla are $|0\rangle$.

First, we take the essential qubit, in some state $|\psi\rangle$ and adjoin to it a prepared ancilla. This is, to begin with, a linear embedding E of the Hilbert space \mathbb{C}^2 for a single qubit into the Hilbert space \mathbb{C}^4 for two qubits; it sends $|\psi\rangle$ to $|0, \psi\rangle$. The embedding preserves lengths of vectors, and we get a channel by sending each linear operator R on \mathbb{C}^2 to the linear operator ERE^\dagger on \mathbb{C}^4 . This is the channel for the first part of our scenario.

Second, we apply the unitary operator U to the two-qubit system. That corresponds to the channel that sends any linear operator R on \mathbb{C}^4 to URU^\dagger .

Finally, we measure the ancilla. The measurement involves two projection operators P_0 and P_1 , from \mathbb{C}^4 to \mathbb{C}^2 , corresponding to the values $|0\rangle$ and $|1\rangle$ for the ancilla respectively. Each P_i induces a linear transformation $T_i(R) = P_i R P_i^\dagger$ from linear operators on \mathbb{C}^4 to linear operators on \mathbb{C}^2 , but the projection operators P_i do not preserve lengths of vectors, and the transformations T_i do not preserve traces. If we discarded the results in the case of failure, we'd have only T_0 , which isn't a channel. But by keeping the (one-qubit) result in both cases, we get $T_0 + T_1 : R \mapsto \sum_{i=0}^1 P_i R P_i^\dagger$, and this is a channel, with $\sum_{i=0}^1 P_i P_i^\dagger = I + I = 2I$. The factor 2 is needed to make the super-operator trace-preserving; the source and target Hilbert spaces have different dimensions.

Composing the three parts, we have a channel

$$R \mapsto \sum_{i=0}^1 (P_i U E) R (P_i U E)^\dagger.$$

6.2. One numerical function of suites. If L is a nonzero linear operator on \mathcal{H} , let $\rho(L)$ be the ratio $\lambda_{\min}/\lambda_{\max}$ of the minimal and maximal eigenvalues of $L^\dagger L$. According to Theorem 2, $\rho(L)$ is the maximum of the guaranteed success probabilities of unitary operators on \mathcal{H}^+ realizing L exactly. Since $\rho(L) = \rho(L')$ if L, L' are collinear, define $\rho(QL) = \rho(L)$. For any exactly realizable suite σ define $\rho(\sigma)$ to be the supremum of $\rho(QL)$ taken over all nonzero linear operators L such that σ extends to QL . Finally, observe that a nonzero linear operator L is singular if and only if $\rho(L) = 0$.

Claim 32. *If σ is a suite in the closure of PL such that, for some fixed $\rho_0 > 0$, every neighborhood of σ contains a PL suite τ with $\rho(\tau) > \rho_0$, then σ belongs to PL .*

An equivalent way to formulate the claim is that, if σ is on the border of PL then every sequence τ_1, τ_2, \dots of PL suites converging to σ must have $\rho(\tau_k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Apply the hypothesis of the claim to choose a sequence τ_1, τ_2, \dots of PL suites converging to σ and having $\rho(\tau_k) > \rho_0$. Choose linear operators L_k with $\rho(L_k) > \rho_0$ such that τ_k extends to QL_k ; we may assume that $\lambda_{\max}(L_k) = 1$, so that $\lambda_{\min}(L_k) > \rho_0$.

Every L_k has a polar decomposition $U_k P_k$ where U_k is unitary and P_k is Hermitian and positive definite (not just semi-definite, because L_k is invertible). Notice that $L_k^\dagger L_k = P_k^\dagger P_k$. Since P_k is Hermitian, it can be diagonalized, say $P_k = B_k D_k B_k^\dagger$ where B_k is unitary and D_k is diagonal. Since the eigenvalues of $P_k^\dagger P_k$ lie between ρ_0 and 1, the diagonal entries in D lie between $\sqrt{\rho_0}$ and 1.

By passing to a subsequence of τ_1, τ_2, \dots , we can arrange that the unitary matrices U_k converge to some unitary matrix U (because the unitary group is compact), that the unitary matrices B_k converge to a unitary matrix B (same reason), that therefore $B_k^\dagger \rightarrow B^\dagger$, that the diagonal matrices D_k converge to a diagonal matrix D (because the eigenvalues all lie in the bounded interval $[\sqrt{\rho_0}, 1]$), and that therefore the matrices P_k converge to a matrix $P = BDB^\dagger$, and the L_k converge to some $L = UP$.

Because of the convergence, we have that the eigenvalues of D lie in $[\sqrt{\rho_0}, 1]$ and, in particular, are positive. So D is invertible, and therefore so are $P = BDB^\dagger$ and $L = UP$.

Finally, let \vec{d} and \vec{r} be the domain and range parts of σ respectively, and let \vec{d}_k and \vec{r}_k be the domain and range parts of τ_k respectively. Since $L_k(\vec{d}_k) = \tau_k(\vec{d}_k) = \vec{r}_k \rightarrow \vec{r}$ as $k \rightarrow \infty$, and as $L_k(\vec{d}) \rightarrow L(\vec{d})$ by continuity, we have $L(\vec{d}) = \vec{r}$. Thus, σ is in PL, as claimed. \square

6.3. Inapproximability in the single-qubit case. According to §5.2, the cross-ratio does not extend continuously to tetrads of points in Riemann sphere that are in configurations 3+1 or 4; any neighborhood of any tetrad in either of these configurations contains general-position tetrads with all possible cross-ratios. Thus there cannot be a theorem of the form: If the cross-ratios of two general-position tetrads differ by at least ε , then the suite of length 8 consisting of these two tetrads cannot be within δ of the FL manifold of suites of length 8. Indeed, no matter how big we make ε and how small we make δ , counterexamples can be found within δ of the double-suite $(0, 0, 0, 0; 0, 0, 0, 0)$.

The best we can hope to do in the direction of such an inapproximability theorem is to assume, as an additional hypothesis, that the

tetrads in question are bounded away from the singular locus of the cross-ratio, i.e., the locus S of tetrads of configurations $3 + 1$ and 4 .

Claim 33. *Let ε and γ be positive real numbers. Then there is a positive real δ with the following property. Let t and t' be tetrads whose distance from the singular locus S of the cross-ratio function is at least γ . Suppose further that the distance between their cross-ratios is at least ε . Then the distance between t and t' is at least δ .*

Proof. Let $\gamma > 0$ be given and let D be the space of tetrads whose distance from S is at least γ . This is a closed subspace of the compact space of all tetrads, so it is also compact. The cross-ratio is a continuous function from D to the Riemann sphere, so, by compactness, it is uniformly continuous. Given $\varepsilon > 0$, let $\delta > 0$ be as in the definition of uniform continuity: Any two points of D whose distance is $< \delta$ have cross-ratios whose distance is $< \varepsilon$. In view of the invariance of the cross-ratio under fractional linear transformations, that is exactly (the contrapositive of) the assertion of the proposition. \square

The preceding argument is valid for any distance functions inducing the usual topologies on the space of tetrads and on the Riemann sphere. Quantitative information about how the δ in the claim varies as a function of γ and ε could be obtained by methods of elementary calculus (Lagrange multipliers).

6.4. Variety for PL in the single-qubit case. Proposition 21 asserts that, in any coordinate patch of $\mathcal{P}^{2\ell}$, there exists an algebraic variety V such that $\text{PL} \subseteq V$ and $G \cap V \subseteq \text{PL}$ for some full-measure open set G . The proof of the proposition is not constructive. We can do better and provide a constructive proof for the proposition. Here we restrict attention to the single qubit case where the construction is especially easy due to the cross-ratio function. In the general case, one can use the construction of proof of Lemma 16.

If $\ell \leq 3$ then, by Proposition 16, every general-position suite belongs to PL, so V could be given by $0 = 0$. Suppose that $\ell \geq 4$.

Let V be the algebraic variety in the Riemann sphere, in variables $a_1, \dots, a_\ell, b_1, \dots, b_\ell$ given by $\ell - 3$ polynomial equations obtained from $\ell - 3$ equations

$$(1) \quad \chi(a_1, a_2, a_3, a_i) = \chi(b_1, b_2, b_3, b_i) \quad \text{where } i = 4, \dots, \ell$$

by clearing fractions.

Claim 34. *$\text{PL} \subseteq V$ and every general-position suite in V belongs to PL .*

Proof. The inclusion $PL \subseteq V$ follows from the fact that fractional linear transformations preserve cross-ratios. Suppose that a general-position suite σ satisfies our polynomial equations. Then it also satisfies the equations (1). By Lemma 16, there is a fractional linear transformation f that sends (a_1, a_2, a_3) to (b_1, b_2, b_3) . For each $i = 4, \dots, \ell$, we have also

$$\begin{aligned} \chi(b_1, b_2, b_3, b_i) &= \chi(a_1, a_2, a_3, a_i) \\ &= \chi(fa_1, fa_2, fa_3, fa_i) \\ &= \chi(b_1, b_2, b_3, fa_i), \end{aligned}$$

which implies that $f(a_i) = b_i$. □

Finally, let's consider suites of length 8 with domain $(0, \infty, 1, -1)$.

Claim 35. *Every PL suite of the form $(0, \infty, 1, -1, a, b, c, d)$ satisfies the equation*

$$(2) \quad \frac{ab + cd}{2} = \frac{a + b}{2} \cdot \frac{c + d}{2}$$

Equation (2) is easy to remember due to the slogan “average of products equals product of averages.”

Proof. First we show that every PL suite of the form $(0, \infty, 1, q, a, b, c, d)$ satisfies the equation

$$(3) \quad (1 - q)(ab + cd) = a(c - qd) + b(d - qc).$$

where q is the inverse of $\chi(a, b, c, d)$.

Indeed, the unique fractional linear transformation sending (a, b, c) to $(0, \infty, 1)$ is

$$z \mapsto \frac{z - a}{z - b} \cdot \frac{c - b}{c - a}$$

So (a, b, c, d) is the image of $(0, \infty, 1, q)$ under a fractional linear transformation if and only if the exhibited transformation sends d to q , i.e., if and only if

$$(4) \quad q = \frac{d - a}{d - b} \cdot \frac{c - b}{c - a}$$

so that q is the inverse of $\chi(a, b, c, d)$. Clearing fractions and rearranging terms in (4), we get equation (3) which yields equation (2) in case $q = -1$. □

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