On Optimal Frame Expansions for Multiple Description Quantization

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Abstract — **We study the problem of finding the optimal overcomplete (frame) expansion and bit allocation for multiple description quantization of a Gaussian signal at high rates over a lossy channel.**

I. Introduction

The setup is shown in Figure 1. In multiple description quantization using overcomplete (frame) expansions [1, 2], an input signal $x \in \mathbb{R}^K$ is represented by a vector $y = Fx \in \mathbb{R}^N$, $N > K$. F is a $N \times K$ matrix, called the frame operator. It is assumed any K rows of F span $\mathbb{R}^K$. The coefficients of y are scalar quantized to obtain $\hat{y}$, and are then independently entropy coded using on average a total of R bits allocated among the N coefficients. In channel state s, the decoder receives $N_{r,s} \leq N$ coefficients after potential erasures, and reconstructs the signal $\hat{x}$ from the received coefficients. The number of channel states is 2^N since each coefficient can be either received or lost. For a given distribution over channel states, we wish to find the frame operator F and the bit allocation for the transform coefficients that minimizes the expected squared error $D = E[\|x - \hat{x}\|^2]$ subject to a constraint on the average rate R, for asymptotically large R and Gaussian x.

II. Analysis

Without loss of generality, assume that x is distributed with zero mean and diagonal covariance matrix $R_{xx} = \text{diag}(\sigma_0^2, ..., \sigma_{K-1}^2)$ (else can use KLT). Let $q = y - \hat{y}$ be the quantization error and let $e = x - \hat{x}$ be the reconstruction error. At high rate, assume q is distributed with zero mean and diagonal covariance matrix with $E[\|q_i\|^2] = c\sigma_{y_i}^2 2^{-2R_i}$, where $c = \pi e/6$ if entropy coded uniform scalar quantization is used. The distortion can be written as $D = \sum_s p_s D_s$, where $D_s = E[\|e\|^2 | S = s]$, and p_s is the probability of the channel being in state s. Let $y_{r,s}$ denote the $N_{r,s}$ dimensional vector of received coefficients. Let $F_{r,s}$ be a $N_{r,s} \times K$ matrix consisting of rows of F corresponding to the received coefficients.

To obtain an expression for D_s, there are two cases to consider: $N_{r,s} \geq K$ and $N_{r,s} < K$. When $N_{r,s} \geq K$, the decoder has enough information to localize the input vector to a finite cell. Although the actual reconstruction will use a consistent reconstruction [1, 3], for analysis purposes, we use the optimal linear reconstruction as $\hat{x} = F_{r,s}^{+}\hat{y}_{r,s}$, where F^{+} is the pseudo-inverse of F. Since $x = F_{r,s}^{+}y_{r,s}$, the conditional distortion can be written as $D_s = E[\|e\|^2 | S = s] = E[\|F_{r,s}^{+}q_{r,s}\|^2]$. When $N_{r,s} < K$, then there is not enough information to localize x to a finite cell. In particular x is bounded in $N_{r,s}$ dimensions and unbounded in $K - N_{r,s}$ dimensions. Thus, $x = F_{r,s}^{+}y_{r,s} + (F_{r,s}^{\perp})^T y_{r,s}^{\perp}$, where the rows of $F_{r,s}^{\perp}$ form an orthonormal basis for the subspace orthogonal to the span of the rows of $F_{r,s}$ and $y_{r,s}^{\perp}$ is a $K - N_{r,s}$ dimensional vector. Now the optimal linear reconstruction is $\hat{x} = F_{r,s}^{+}\hat{y}_{r,s} + (F_{r,s}^{\perp})^T E[y_{r,s}^{\perp} | y_{r,s} = \hat{y}_{r,s}]$ which gives a distortion of $D_s = E[\|F_{r,s}^{+}q_{r,s}\|^2] + E[\|y_{r,s}^{\perp}\|^2 | y_{r,s} = \hat{y}_{r,s}]$. Since the source is Gaussian, $E[\|y_{r,s}^{\perp}\|^2 | y_{r,s}]$ can be easily computed.

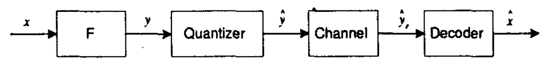

Fig. 1: System setup.

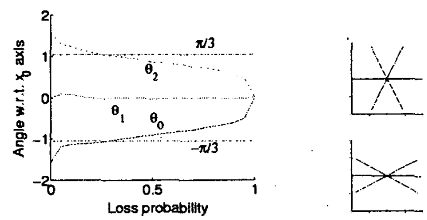

Fig. 2: Results for optimal 3×2 expansion: (a) θ_i (b) φ_i for loss probabilities of 0.2 (top) and 0.95 (bottom).

Using the equations for D_s and the fact that $E[qq^T]$ is diagonal, the portion of distortion that can be minimized by bit allocation can be written as $D_b = \sum_{i=0}^{N-1} \alpha_i \sigma_{y_i}^2 2^{-2R_i}$, where α_i is a function of the transform F, the channel state probabilities p_s, and the quantization constant c. Let D_{nb} be the remaining portion of the distortion D. Minimizing D_b is a classic bit allocation problem with solution given by $R_i = R/N + \log_2(\alpha_i \sigma_{y_i}^2 / (\prod_{j=0}^{N-1} \alpha_j \sigma_{y_j}^2)^{1/N})/2$. This gives an optimal D_b of $D_b^* = N(\prod_{j=0}^{N-1} \alpha_j \sigma_{y_j}^2)^{1/N} 2^{-2R/N}$. To find the optimal transform, we have to minimize $D_b^* + D_{nb}$. Since it is hard theoretically, we use numerical gradient descent techniques by varying one coefficient at a time.

Results show that at high loss rates D_{nb} is the dominating term which is minimized by repeating the coefficient with highest variance. At low loss rates, D_b^* is the dominating term which is minimized by the optimal source coder. Results are shown for 3×2 expansion in Figure 2, where the values for $\theta_i = \tan^{-1}(F_{i1}/F_{i0})$, $i = 0, 1, 2$ are plotted with rate constraint $R = 6$ bits and variances $\sigma_0^2 = 4$ and $\sigma_1^2 = 1$. Also shown is φ_i, which is the ith row of matrix F.

References

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