

U-Prove Range Proof Extension

Draft Revision 1

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Summary

This document extends the U-Prove Cryptographic Specification [\[UPCS\]](#) by specifying set membership proofs. This allows proving that a committed value is less than, less than or equal to, greater than, or greater than or equal to another (committed) value.

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Change history

Version	Description
Revision 1	Initial draft

1 Introduction

This document extends the U-Prove Cryptographic Specification [\[UPCS\]](#) by specifying range proofs. The Prover will prove to the Verifier that a committed value is less than, less than or equal to, greater than, or greater than or equal to another (committed) value.

The Prover knows a secret value a , and will prove to the Verifier an inequality relation between a and another value b that may or may not be known to the Verifier. The Prover and Verifier have as common input a pair of generators $g, h \in G_q$. The Prover will create one of the following proofs:

$$\pi_{\odot} = PK\{\alpha, \beta, \gamma, \delta | C_A = g^{\alpha} h^{\gamma} \cap C_B = g^{\beta} h^{\delta} \cap \alpha \odot \beta\}$$

or

$$\pi_{\odot} = PK\{\alpha, \gamma | C_A = g^{\alpha} h^{\gamma} \cap \alpha \odot b\}$$

where $\odot \in \{<, \leq, >, \geq\}$. The Prover knows assignments for $(\alpha, \beta, \gamma, \delta)$.

The proof relies on comparing the bit decompositions of a and b . The Prover computes Pedersen commitments to the bit decompositions and then proves they are formed correctly. Then, the Prover compares each i bit prefix of a and b ; the results of the comparisons are stored in helper commitments D_i . The Prover creates an Equality Proof to show that the D_i are computed correctly. The committed value in D_{n-1} is equal to $\{-1, 0, 1\}$ depending on the relationship between a and b . The Prover adds an auxiliary proof showing that the committed value in D_{n-1} is equal to the appropriate value given \odot .

The U-Prove Cryptographic Specification [\[UPCS\]](#) allows the Prover, during the token presentation protocol, to create a Pedersen Commitment and show that the committed value is the equal to a particular token attribute. The Prover MAY use this Pedersen Commitment as either C_A or C_B . The Issuance and Token Presentation protocols are unaffected by this extension. The Prover may choose to create a range proof after these two protocols complete.

The committed values in C_A and C_B MUST NOT be hashed. If any of these values are U-Prove token attributes, the attributes also MUST NOT be hashed.

The Range Proof protocol makes use of the following U-Prove Extensions: Set Membership Proof Extension [\[EXSM\]](#), Bit Decomposition Extension [\[EXBD\]](#), and Equality Proof Extension [\[EXEQ\]](#).

1.1 Notation

In addition to the notation defined in [\[UPCS\]](#), the following notation is used throughout the document. The range proof consists of many sub-protocols; local variables are omitted from this list unless they consistently appear with the same meaning/value.

a Value to be compared to b , known only to Prover.

b Value to be compared to a , MAY be known to Verifier.

C_A Pedersen Commitment to a . IOnly Prover knows opening.

C_B Pedersen Commitment to b , or null if Verifier knows b .

$bIsKnown$ True if Verifier knows b .

$\odot, proofType$	A value in the set $\{<, \leq, >, \geq\}$ indicating the relationship between a and b that needs to be proven.
min	Minimum possible value for a and b .
max	Maximum possible value for a and b .
\mathcal{M}	An equality map, as defined in U-Prove Equality Proof Extension [EXEQ]. Range proofs require multiple different equality maps; this document uses local variable \mathcal{M} to refer to a map.
\bar{A}_i	The value of a DL Equation, as defined in U-Prove Equality Proof Extension [EXEQ]. Range proofs create multiple different equality proofs; this document uses local variable \bar{A}_i to refer to the DL Equation values.
$\bar{g}_{i,j}$	The bases of a DL Equation, as defined in U-Prove Equality Proof Extension [EXEQ]. Range proofs create multiple different equality proofs; this document uses local variable $\bar{g}_{i,j}$ to refer to the DL Equation bases.
$\bar{x}_{i,j}$	The witnesses (exponents) for a DL Equation, as defined in U-Prove Equality Proof Extension [EXEQ]. Range proofs create multiple different equality proofs; this document uses local variable $\bar{x}_{i,j}$ to refer to the DL Equation witnesses.
$\vec{a} = (a_0, r_0), (a_1, r_1), \dots, (a_{n-1}, r_{n-1})$	The opening information for Pedersen Commitments \vec{A} . The a_i contain the bit decomposition of $a - min$, while the r_i are the second exponent.
$\vec{b} = (b_0, s_0), (b_1, s_1) \dots, (b_{n-1}, s_{n-1})$	The opening information for Pedersen Commitments \vec{B} . The b_i contain the bit decomposition of $b - min$, while the s_i are the second exponent. If the Verifier knows b , then $s_i = 0$.
$\vec{c} = (c_0, y_0), (c_1, y_1) \dots, (c_{n-1}, y_{n-1})$	The opening information for Pedersen Commitments \vec{C} . The c_i contain the difference between \vec{a} and \vec{b} : $c_i = a_i - b_i$, while the y_i are the second exponent.
$\vec{d} = (d_1, t_1) \dots, (d_{n-1}, t_{n-1})$	The opening information for Pedersen Commitments \vec{D} . Each d_i stores the inequality relationship between the i least significant bits of a and b , represented as a value in $\{-1, 0, 1\}$. The t_i are the second exponent.
$\vec{e} = (e_1, v_1) \dots, (e_{n-1}, v_{n-1})$	The opening information for Pedersen Commitments \vec{E} . Each e_i is actually equal to d_{i-1} , while the v_i are the second exponent.
$\vec{x} = (c_1, m_1) \dots, (c_{n-1}, m_{n-1})$	The opening information for Pedersen Commitments \vec{X} . Each c_i is actually equal to the c_i in \vec{c} , while the m_i are the second exponent.
$\vec{A} = A_0, A_1, \dots, A_{n-1}$	Pedersen Commitments to \vec{a} .
$\vec{B} = B_0, B_1, \dots, B_{n-1}$	Pedersen Commitments to \vec{b} .
$\vec{C} = C_0, C_1, \dots, C_{n-1}$	Pedersen Commitments to \vec{c} .
$\vec{D} = D_1, \dots, D_{n-1}$	Pedersen Commitment to \vec{d} .
$\vec{E} = E_1, \dots, E_{n-1}$	Pedersen Commitment to \vec{e} .
$\vec{X} = X_1, \dots, X_{n-1}$	Pedersen Commitment to \vec{x} .
π_A	Proof that \vec{A} is a valid commitment to the bit decomposition of $a - min$.
π_B	Proof that \vec{B} is a valid commitment to the bit decomposition of $b - min$. Null if the Verifier knows b .

π_C Main equality proof showing that \vec{D} and \vec{X} are formed correctly.

π_D Auxiliary proof showing that D_{n-1} contains the correct value; either and equality proof or a set membership proof.

$a \leftarrow A$ Choose a uniformly at random from set A .

The key words “MUST”, “MUST NOT”, “SHOULD”, “RECOMMENDED”, “MAY”, and “OPTIONAL” in this document are to be interpreted as described in [\[RFC 2119\]](#).

1.2 Feature overview

The Prover knows the opening of a Pedersen Commitments $C_A = g^a h^r$ and $C_B = g^b h^s$ (optionally, b may be public knowledge). The Prover needs to show that the relationship $a \odot b$ holds, where $\odot \in \{<, \leq, >, \geq\}$ is also known to the Verifier. For efficiency, the Prover and Verifier both know that a and b fall inside the range $[min, max]$. The Prover will create a special-honest verifier zero-knowledge proof of knowledge that the Prover knows a tuple of values (a, r, b, s) such that:

- 1 $C_A = g^a h^r$.
- 2 $C_B = g^b h^s$.
- 3 The relationship $a \odot b$ holds, where $\odot \in \{<, \leq, >, \geq\}$.

The range proof consists of the following components:

1. Pedersen commitments A_0, A_1, \dots, A_{n-1} to the bit decomposition of $a - min$, as well as a Bit Decomposition Proof [\[EXBD\]](#) showing the A_i are constructed correctly.
2. (Optional) Pedersen commitments B_0, B_1, \dots, B_{n-1} to the bit decomposition of $b - min$, as well as a Bit Decomposition Proof [\[EXBD\]](#) showing the B_i are constructed correctly.
3. Pedersen commitments X_0, \dots, X_{n-1} to $c_i = (a_i - b_i)^2$. These are helper values
4. Pedersen commitments D_1, \dots, D_{n-1} to $d_i \in \{-1, 0, 1\}$, which represents the inequality relationship between the i least significant bits of a and b . We compute it as follows:

$$d_i = \begin{cases} a_i - b_i & i = 0 \\ d_{i-1} - d_{i-1}(a_i - b_i)^2 + (a_i - b_i) & i > 0 \end{cases}$$

5. An Equality Proof [\[EXEQ\]](#) showing the X_i and D_i are formed correctly.
6. An auxiliary proof showing that D_{n-1} is a commitment to the appropriate value in $\{-1, 0, 1\}$ given the type of inequality relationship the Prover is trying to prove.

2 Protocol specification

As the range proof can be performed independently of the U-Prove token presentation protocols, the common parameters consist simply of the group G_q , two generators g and h , and a cryptographic function \mathcal{H} . The commitments C_A and C_B MAY be generated by the Prover.

The remaining parameters may be chosen by either the Prover or Verifier: The values min and max indicate the maximum span for secret values a and b . The variable $bIsKnown$ indicates whether the Verifier knows b . The $proofType$ indicates the inequality relationship between a and b that the Prover wishes to demonstrate.

2.1 Common Protocols

The main body of the range proof is an Equality Proof defined in the U-Prove Equality Proof Extension [\[EXEQ\]](#). `EQProofParams()` returns the common parameters for the main proof. It generates an equality map \mathcal{M} and sets up the DL equations $\bar{A}_i = \prod_{j=0}^{n_i-1} \bar{g}_{i,j}^{\alpha_{i,j}}$ where the \bar{A}_i and $\bar{g}_{i,j}$ are public values returned by this protocol, while the $\alpha_{i,j}$ are secret values known only to the Prover.

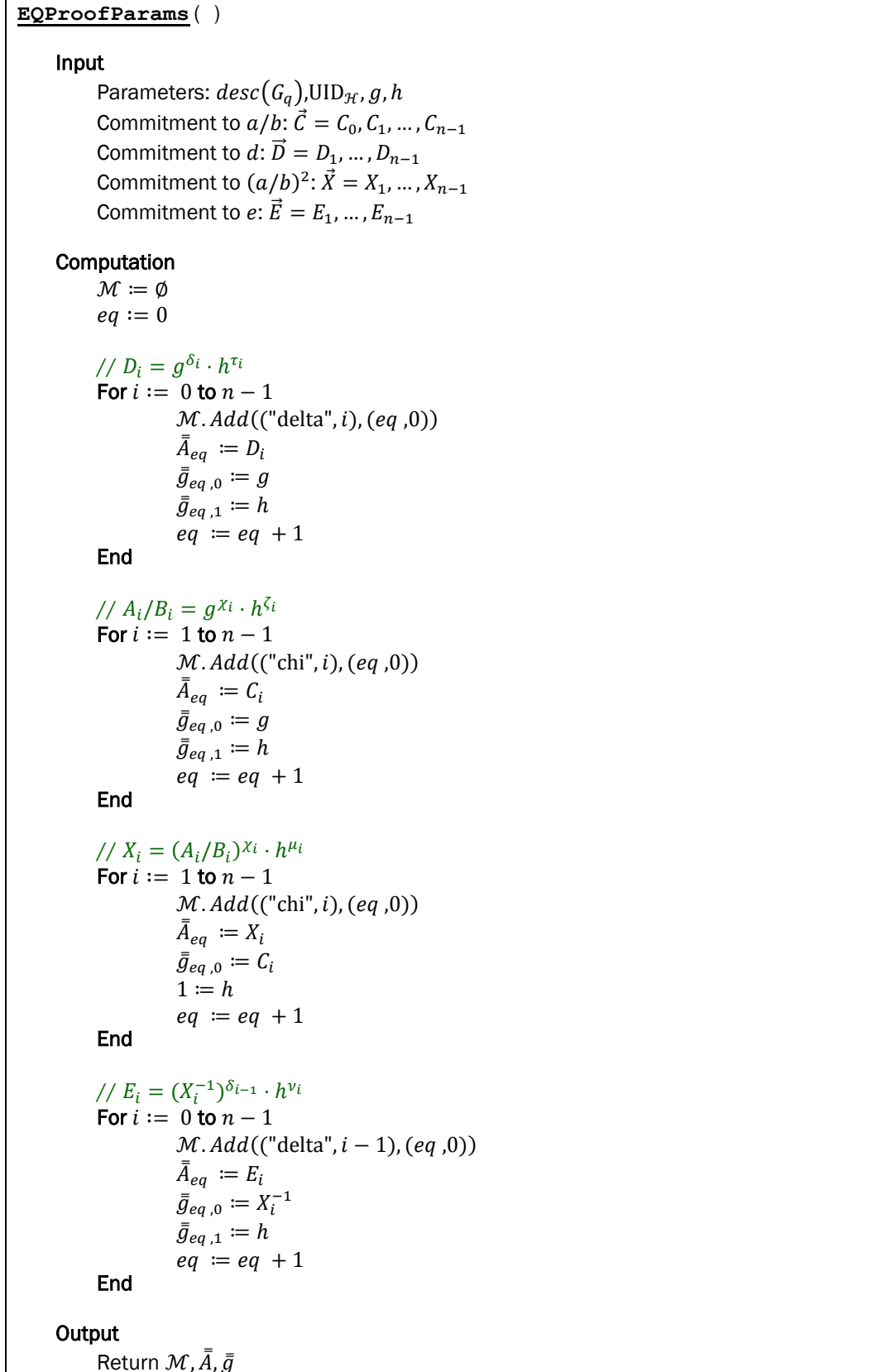


Figure 1: EQProofParams.

2.2 Presentation

The Prover calls RangeProve to generate a range proof. We break up the range proof presentation protocol into various sub-protocols for ease of exposition. The range proof also requires calling protocols from Bit Decomposition Proof [EXBD], Set Membership Proof [EXSM], and Equality Proof [EXEQ].

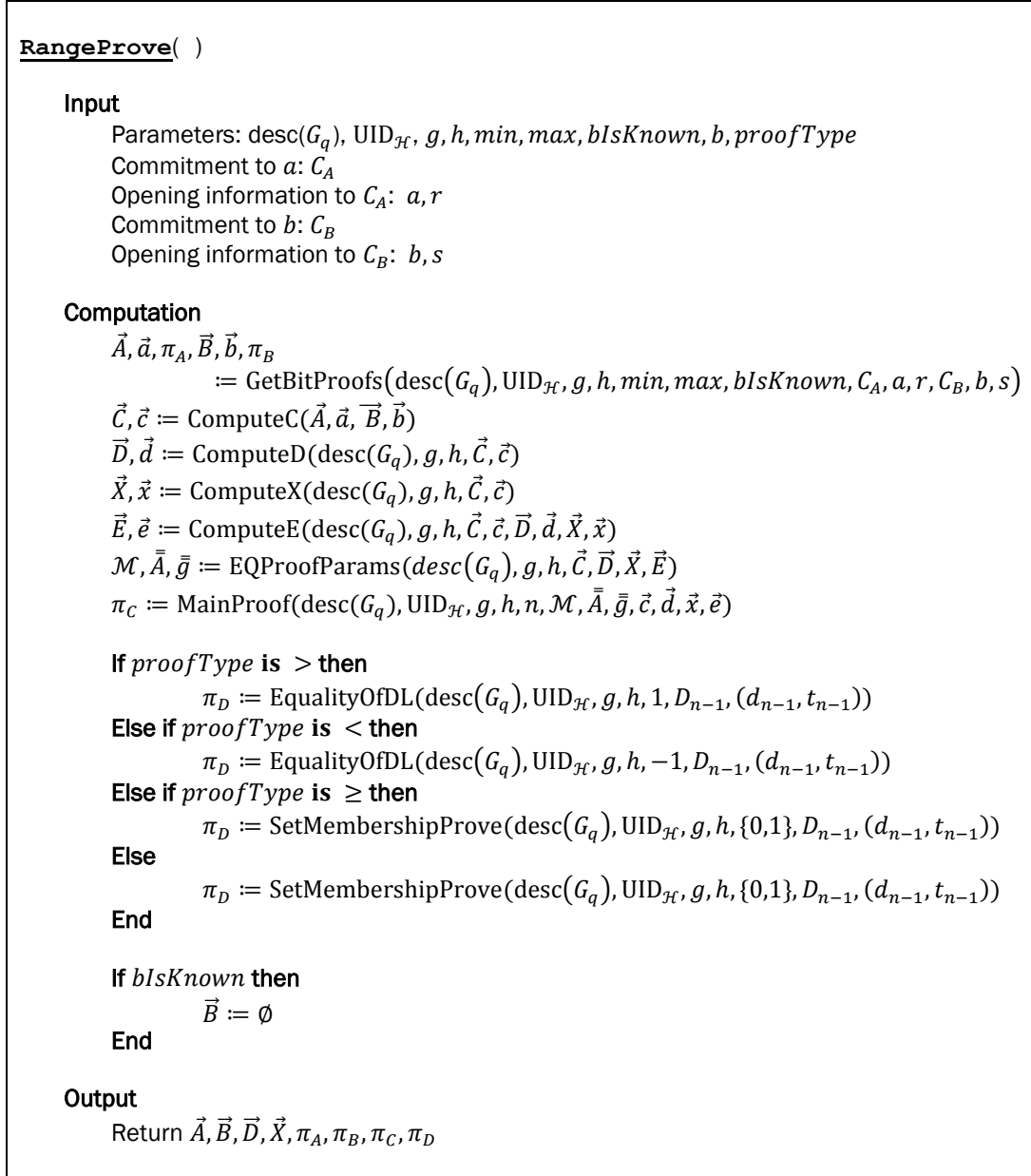


Figure 2: RangeProve

The range proof requires dividing the bit decomposition of A by the bit decomposition of B to get an array of Pedersen commitments \vec{C} and their openings \vec{c} . This step is performed in the function `ComputeC()`.

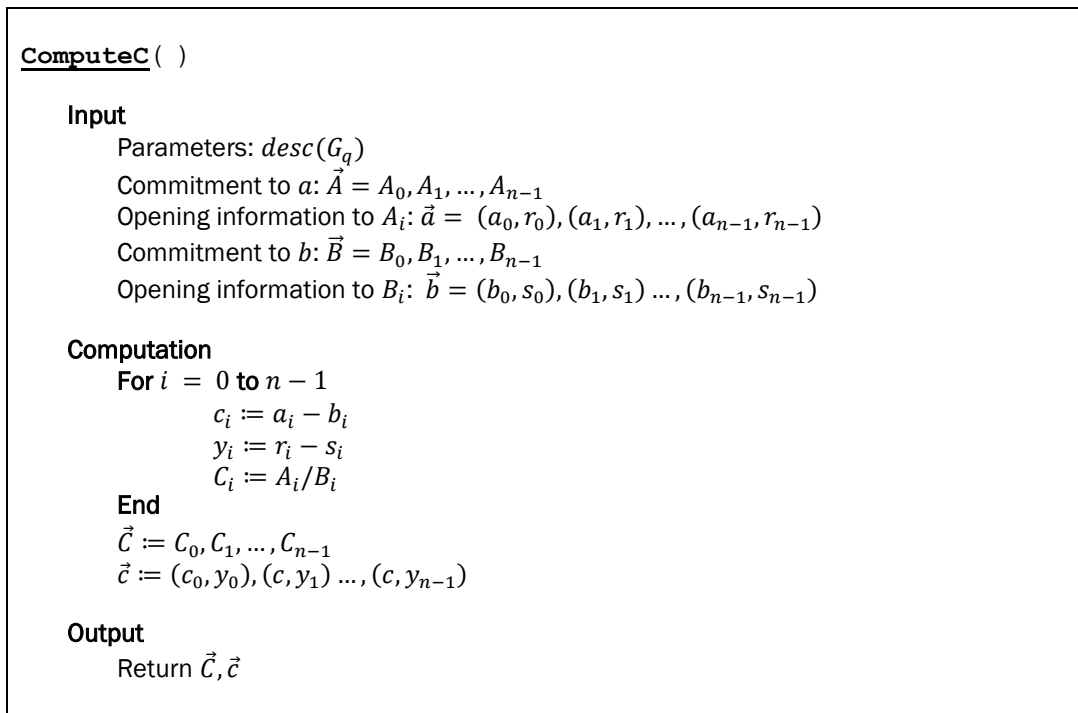


Figure 3: ComputeC

The range proof performs bit decompositions of a and b with the help of protocols from U-Prove Bit Decomposition Extension [EXBD]. For efficiency, it normalizes the range from $[min, max]$ to $[0, max - min]$. This step is important since the length of the range proof depends on the length of the bit decomposition. If the value of b is known to the Verifier, the Prover will generate a default Pedersen Commitments to the bit decomposition of b and omit the bit decomposition proof.

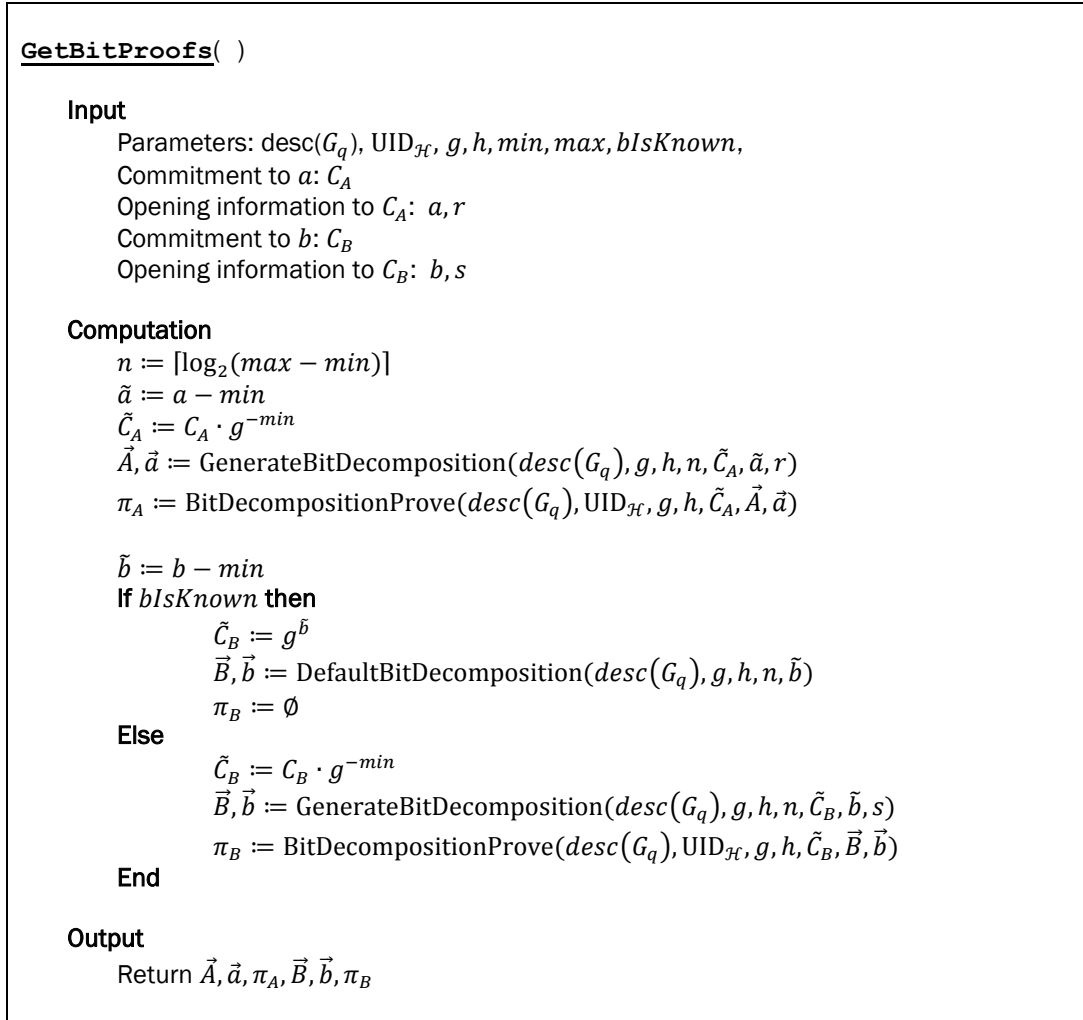


Figure 4: GetBitProofs.

The following two protocols generate a bit decomposition of an integer x and return Pedersen Commitments and their openings to this decomposition. `GenerateBitDecomposition()` generates random Pedersen Commitments, while `DefaultBitDecomposition()` sets the second exponent to 0.

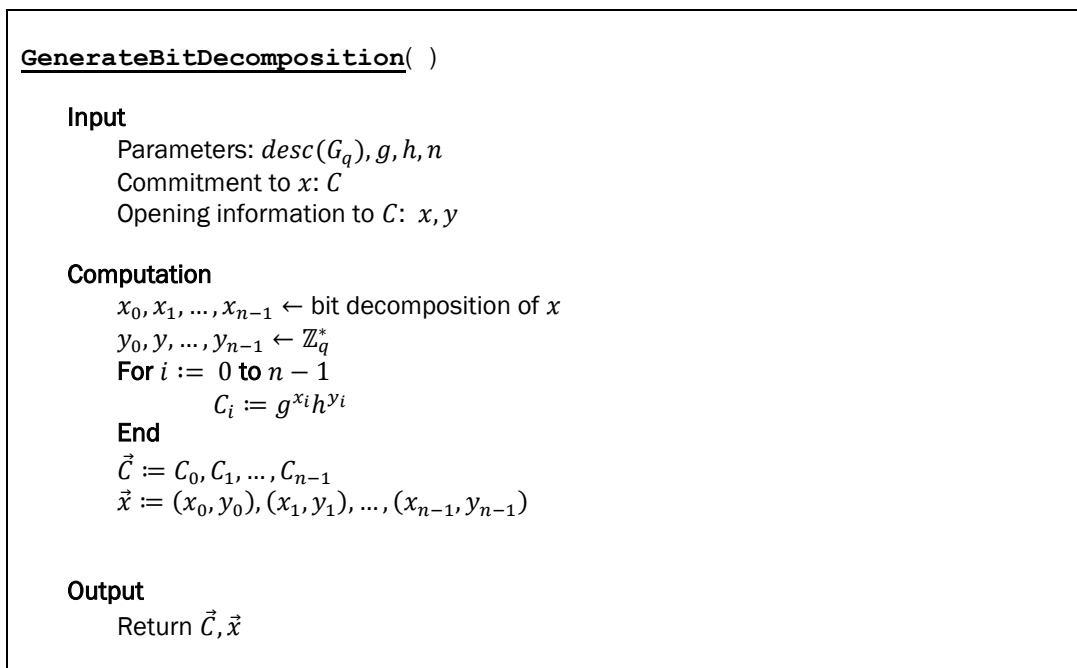


Figure 5: GenerateBitDecomposition

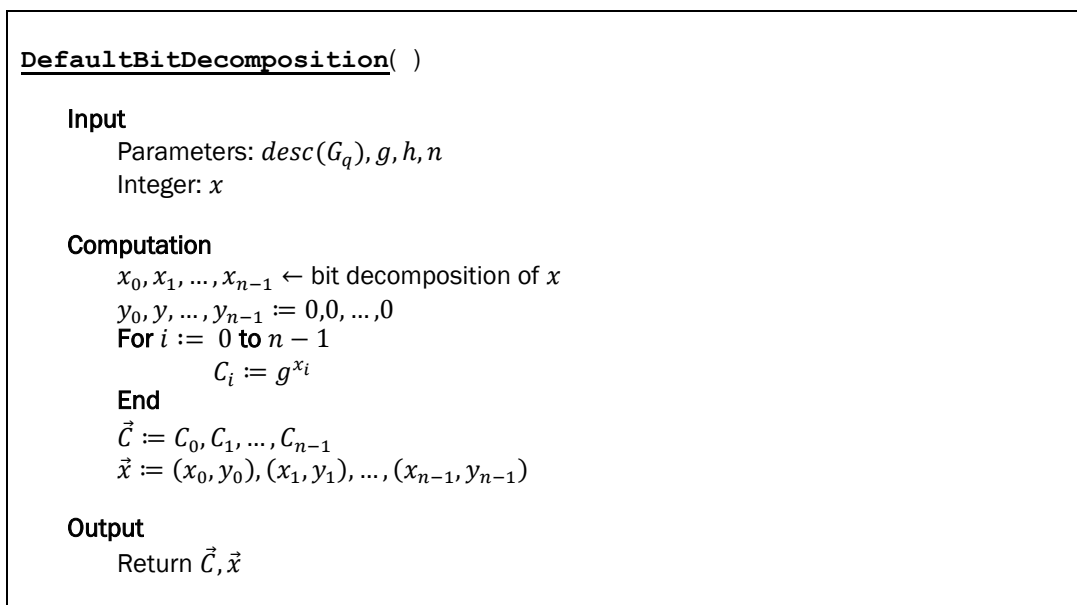


Figure 6: DefaultBitDecomposition

The range proof compares A to B bit by bit. It does so by computing Pedersen commitments D_1, \dots, D_{n-1} to $d_i \in \{-1, 0, 1\}$, which represents the inequality relationship between the i least significant bits of a and b . We compute the d_i as follows:

$$d_i = \begin{cases} a_i - b_i & i = 0 \\ d_{i-1} - d_{i-1}(a_i - b_i)^2 + (a_i - b_i) & i > 0 \end{cases}$$

The function ComputeD() takes as input $c_i = a_i - b_i$, which is substituted into the above formula.

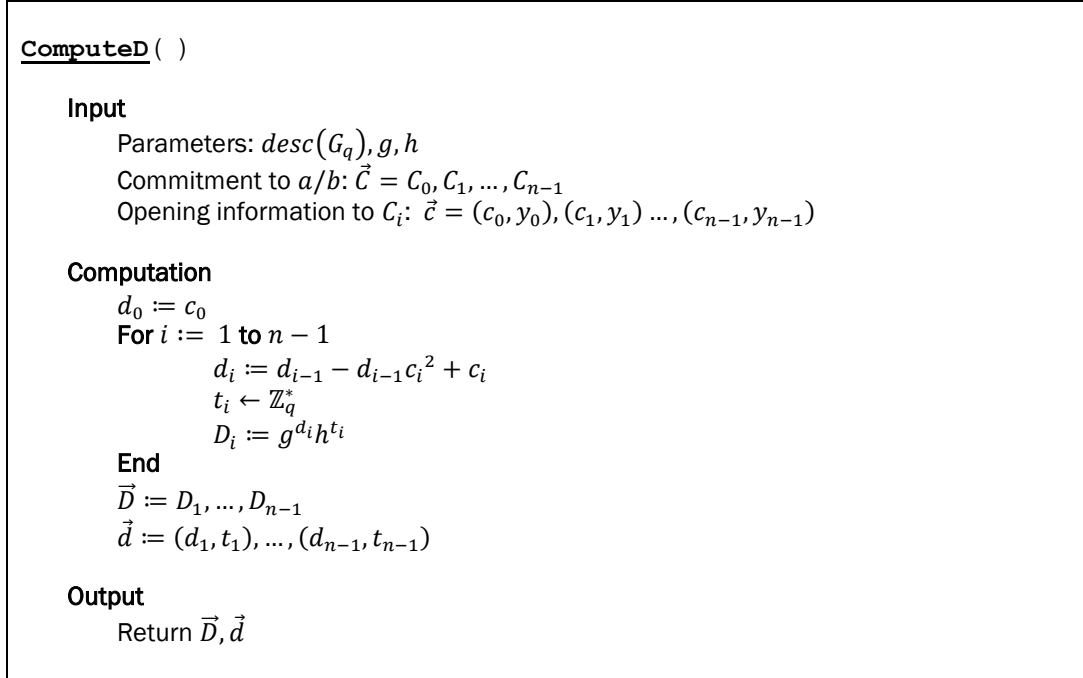


Figure 7: ComputedD.

Proving that the D_i are formed correctly requires helper values $X_i = C_i^{c_i}h^{m_i}$.

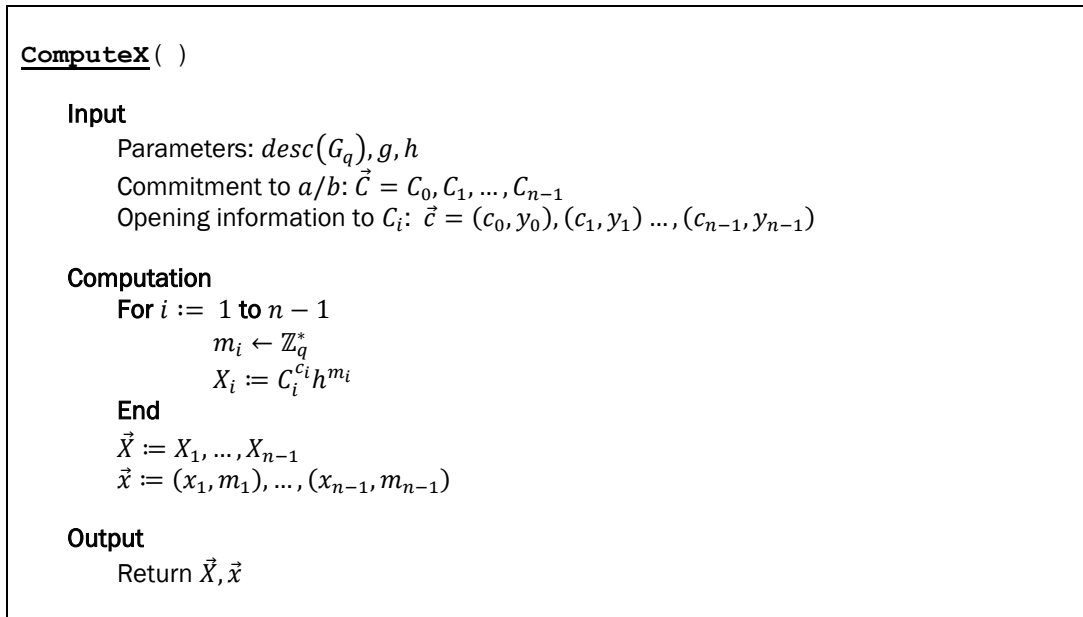


Figure 8: ComputeX.

Proving that the D_i are formed correctly also requires helper values $E_i = (X_i^{-1})^{d_{i-1}} h^{v_i} = D_i \cdot (D_{i-1})^{-1} \cdot (C_i)^{-1}$.

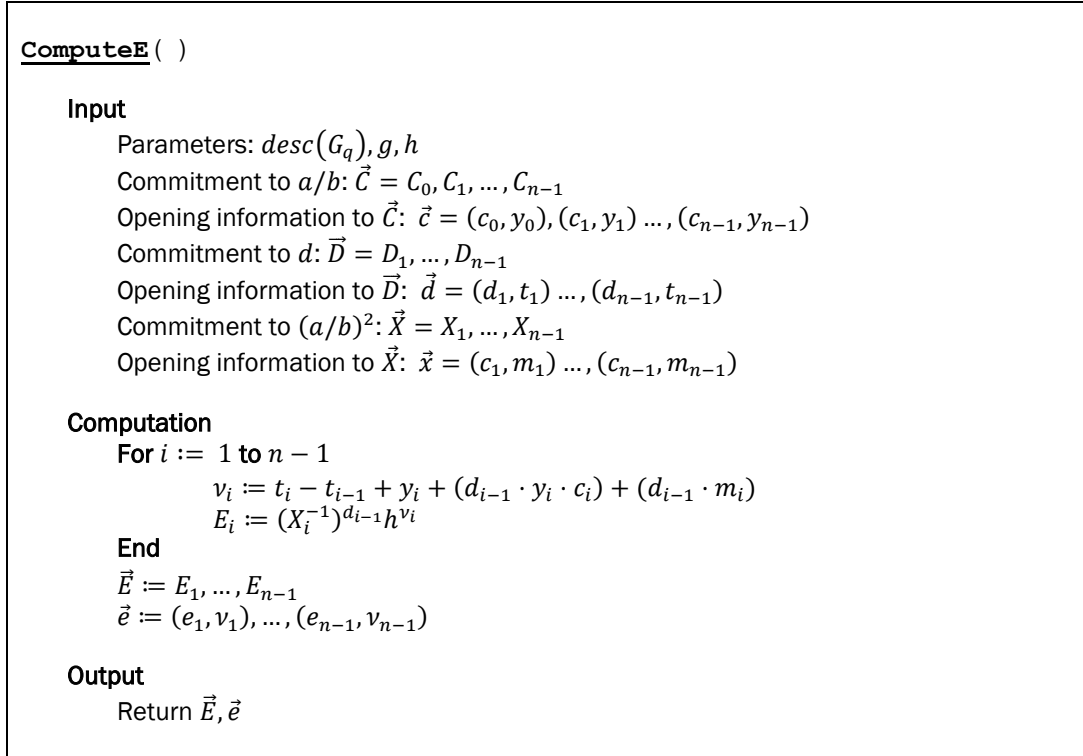


Figure 9: ComputeE

The main body of the range proof is an Equality Proof [EXEQ] showing that $\vec{D}, \vec{X}, \vec{E}$ are formed correctly.

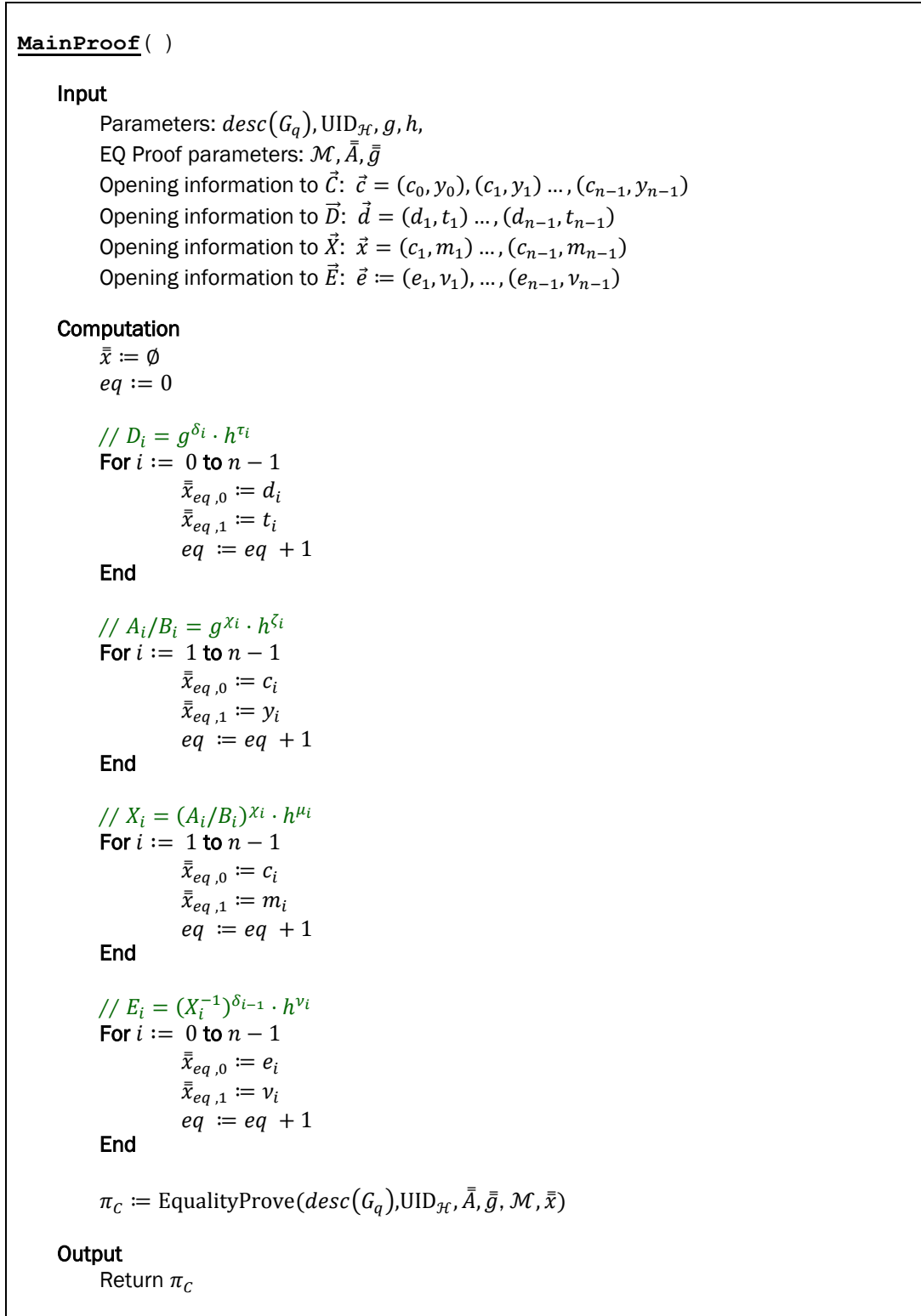


Figure 10: MainProof.

EqualityOfDL is a small helper proof that shows that $D = g^d \cdot h^t$ is a Pedersen Commitment to some integer x known to the Verifier. The protocol generates an Equality Proof [\[EXEQ\]](#).

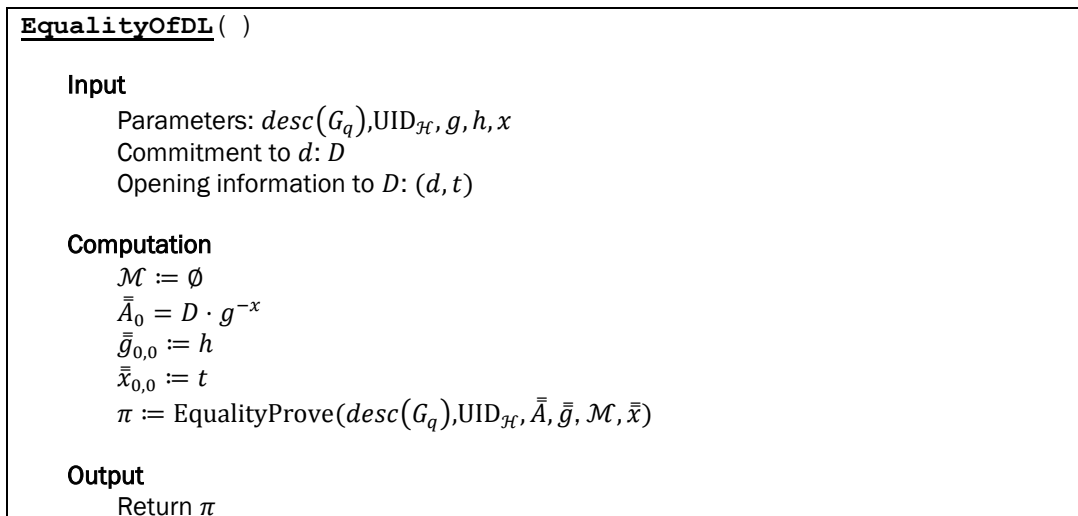


Figure 11: EqualityOfDL.

2.3 Verification

The Verifier receives the common parameters, as well as commitments to a and b and the proof. The Verifier returns true if the verification passes, false otherwise. Verification requires checking the bit decomposition proofs π_A and π_B , the main equality proof π_C , and the auxiliary proof π_D that depends on the proof type.

```

RangeVerify( )

Input
  Parameters: desc( $G_q$ ), UID $_{\mathcal{H}}$ ,  $g$ ,  $h$ ,  $min$ ,  $max$ ,  $bIsKnown$ ,  $b$ ,  $proofType$ 
  Commitment to  $a$ :  $C_A$ 
  Commitment to  $b$ :  $C_B$ 
  Proof:  $\vec{A}, \vec{B}, \vec{D}, \vec{X}, \pi_A, \pi_B, \pi_C, \pi_D$ 

Computation
   $P := true$ 
   $P := P$  AND BitDecompositionVerify(desc( $G_q$ ), UID $_{\mathcal{H}}$ ,  $g$ ,  $h$ ,  $C_A/g^{min}$ ,  $\vec{A}, \pi_A$ )
  if  $bIsKnown$  then
     $\vec{B}, \vec{b} :=$  DefaultBitDecomposition(desc( $G_q$ ),  $g$ ,  $h$ ,  $n$ ,  $b - min$ )
  Else
     $P := P$  AND BitDecompositionVerify(desc( $G_q$ ), UID $_{\mathcal{H}}$ ,  $g$ ,  $h$ ,  $C_B/g^{min}$ ,  $\vec{B}, \pi_B$ )
  End

   $\vec{C} :=$  ComputeClosedC(desc( $G_q$ ),  $\vec{A}, \vec{B}$ )
   $\vec{E} :=$  ComputeClosedE(desc( $G_q$ ),  $\vec{D}, \vec{C}$ )
   $\mathcal{M}, \vec{A}, \vec{g} :=$  EQProofParams(desc( $G_q$ ),  $g$ ,  $h$ ,  $\vec{C}, \vec{D}, \vec{X}, \vec{E}$ )
   $P := P$  AND EqualityVerify(desc( $G_q$ ), UID $_{\mathcal{H}}$ ,  $\vec{A}, \vec{g}, \mathcal{M}, \pi_C$ )

  if  $proofType$  is  $>$  then
     $P := P$  AND EqualityOfDLVerify(desc( $G_q$ ), UID $_{\mathcal{H}}$ ,  $g$ ,  $h$ ,  $D_{n-1}, 1, \pi_D$ )
  Else if  $proofType$  is  $<$  then
     $P := P$  AND EqualityOfDLVerify(desc( $G_q$ ), UID $_{\mathcal{H}}$ ,  $g$ ,  $h$ ,  $D_{n-1}, -1, \pi_D$ )
  Else if  $proofType$  is  $\geq$  then
     $P := P$  AND SetMembershipVerify(desc( $G_q$ ), UID $_{\mathcal{H}}$ ,  $g$ ,  $h$ ,  $D_{n-1}, \{0,1\}, \pi_D$ )
  Else
     $P := P$  AND SetMembershipProve(desc( $G_q$ ), UID $_{\mathcal{H}}$ ,  $g$ ,  $h$ ,  $D_{n-1}, \{-1,0\}, \pi_D$ )
  End

Output
  Return  $P$ 

```

The Verifier uses the function ComputeClosedC() to compute $C_i = A_i/B_i$, which are needed to verify π_C .


```

ComputeClosedC( )

Input
  Parameters:  $desc(G_q)$ 
  Commitment to  $a$ :  $\vec{A} = A_0, A_1, \dots, A_{n-1}$ 
  Commitment to  $b$ :  $\vec{B} = B_0, B_1, \dots, B_{n-1}$ 

Computation
  For  $i := 0$  to  $n - 1$ 
     $C_i := A_i/B_i$ 
  End
   $\vec{C} := C_0, C_1, \dots, C_{n-1}$ 

Output
  Return  $\vec{C}$ 

```

The Verifier calls function `ComputeClosedE()` to compute $E_i = D_i \cdot (D_{i-1})^{-1} \cdot C_i^{-1}$, which are needed to verify π_C .

```

ComputeClosedE( )

Input
  Parameters:  $desc(G_q)$ 
  Commitment to  $d$ :  $\vec{D} = D_1, \dots, D_{n-1}$ 
  Commitment to  $b$ :  $\vec{C} = C_0, C_1, \dots, C_{n-1}$ 

Computation
   $D_0 := C_0$ 
  For  $i := 1$  to  $n - 1$ 
     $E_i := D_i \cdot (D_{i-1})^{-1} \cdot C_i^{-1}$ 
  End
   $\vec{E} := E_0, E_1, \dots, E_{n-1}$ 

Output
  Return  $\vec{E}$ 

```

The Verifier calls `EqualityOfDLVerify` to check that D is a Pedersen Commitment to x .

EqualityOfDLVerify ()	
Input	Parameters: $desc(G_q), UID_{\mathcal{H}}, g, h, x$ Commitment to d : D Proof: π
Computation	$\mathcal{M} := \emptyset$ $\bar{A}_0 := D \cdot g^{-x}$ $\bar{g}_{0,0} := h$ $pass := \text{EqualityVerify}(desc(G_q), UID_{\mathcal{H}}, \bar{A}, \bar{g}, \mathcal{M}, \pi)$
Output	Return $pass$

3 Security Considerations

The range proof invokes protocols from U-Prove Equality Proof Extension [EXEQ], U-Prove Bit Decomposition Extension [EXBD], and U-Prove Set Membership Proof Extension [EXSM]. Its security relies on their security. The following restriction apply:

- The Prover and the Verifier MUST NOT know the relative discrete logarithm $\log_g h$ of the generators g and h . This is not an issue if the generators are chosen from the list of U-Prove recommended parameters.

References

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