A Game Theoretic Model for the Formation of Navigable Small-World Networks

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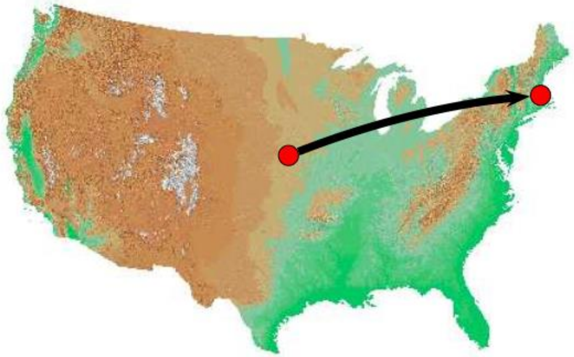
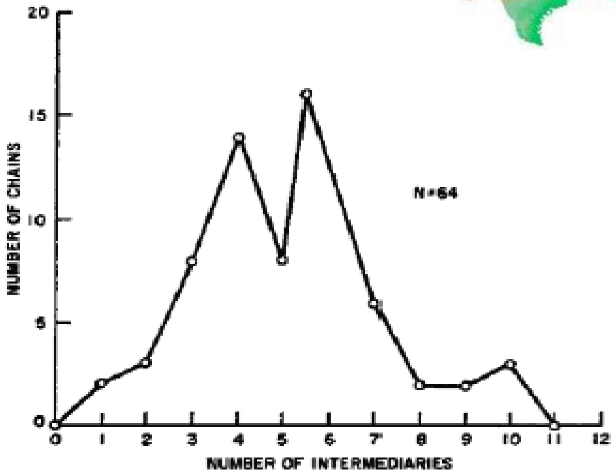
Milgram'67: Six Degrees of Separation

- 296 People in Omaha, NE, were given a letter, asked to try to reach a stockbroker in Boston, MA, via personal acquaintances
- 20% reached target
- average number of "hops" in the completed chains = 6.5
- Why are chains so short?

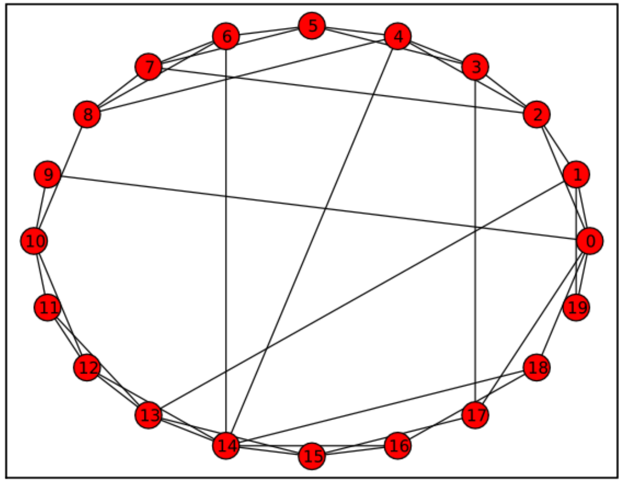




Milgram experiment (Travers-Milgram 1970)

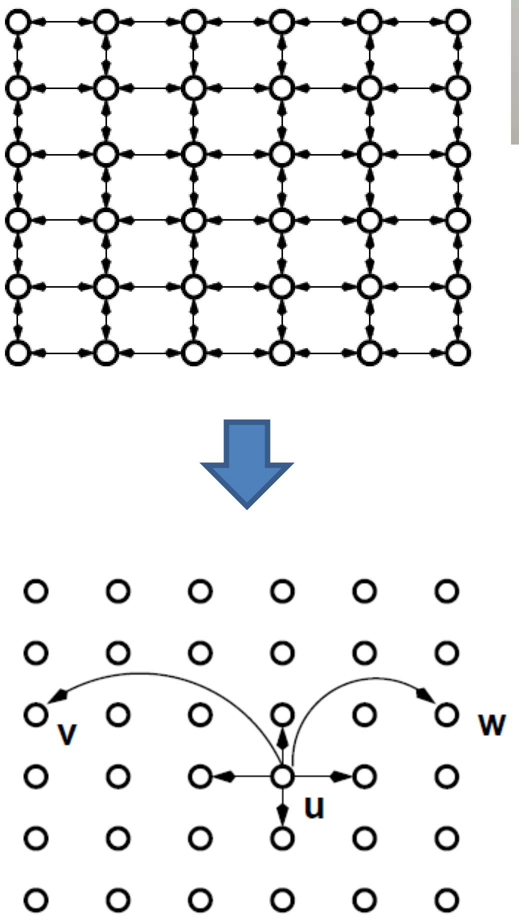
Watts & Strogatz'98: Small-World Model

- Propose two important features of the small-world networks
  - Low diameter
  - High clustering
- Propose a random rewiring model
- But one feature of the Milgram experiment is missing!



Kleinberg'00: Navigable Small World

- Notice the feature of efficient decentralized navigation in Milgram's experiment --- navigability
  - Subjects only use local information to navigate the network
- Adjust the rewiring model of Watts&Strogatz
- Prove the navigability of the model at a critical parameter setting

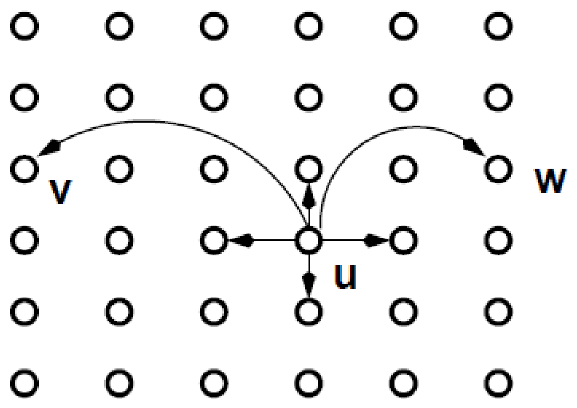


Kleinberg's Small-World Model

- Put n^k people on a k-dimensional grid
- Connect each to its immediate grid neighbors
- Add one directed long-range link per node
  - Node u connects to v with probability

$$\Pr(u \to v) \propto \frac{1}{d(u,v)^r}$$

  - $r \in [0, +\infty)$ is connection preference:
    - r close to ∞: prefers to connect to nodes in the vicinity
    - r close to 0: prefer to connect to faraway nodes equally as neighboring nodes
    - $r = 0$: reduces to the Watts & Strogatz model (random network)

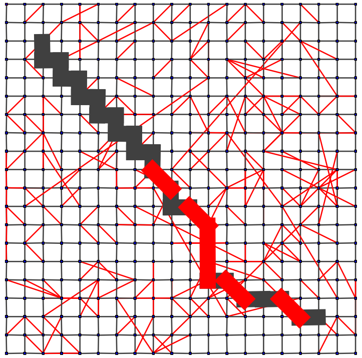


Decentralized Routing in Kleinberg's Model

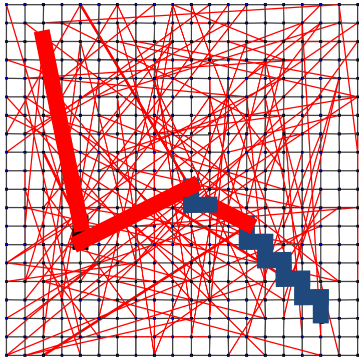
- Decentralized greedy routing:
  - given a target t, every node u routes the message for t to u's neighbor (local or long-range contact) closest to t in grid distance
- Main result:
  - $r = k$: routing is efficient $O(\log^2 n)$ --- navigable network
  - $r < k$ or $r > k$: routing is not efficient $\Omega(n^c)$ for some c related to r.
- Intuition:
  - $r > k$: long-range links are too close to move towards the target
  - $r < k$: long-range links are too random to zoom into the target
  - $r = k$: right balance between fast moving towards and zooming into the target

What is the parameter in real networks?

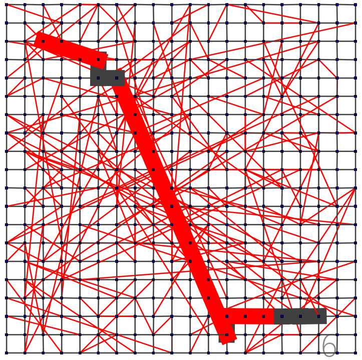
$r > k$



$r < k$

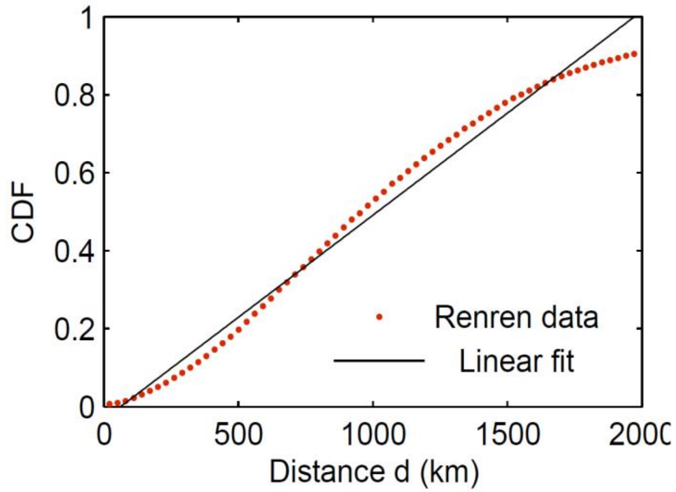


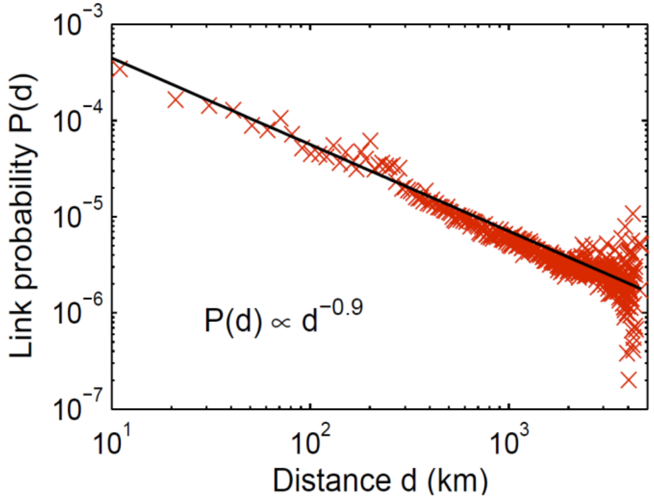
$r = k$



Empirical Validation

- Liben-Nowell et al. '05:
  - fractional dimension α for non-uniform population distribution
    $$|\{w: d(u,w) \leq d(u,v)\}| = c \cdot d(u,v)^{\alpha}$$
  - When $r = \alpha$, the network is navigable
  - LiveJournal dataset (495K nodes): $\alpha \approx 0.8, r = 1.2$
- Ours:
  - Renren dataset (10 mil nodes)
  - $\alpha \approx 1.0, r = 0.9$
- Others show similar results





Why is connection preference close to the critical value of grid dimension in the real world !?

Our Proposal

- Game-theoretic formation of navigable small world
  - strong theoretical and empirical support
  - Navigable small world network is not only one equilibrium, but is the only one tolerating both collusions and random perturbations
  - Surprising connection with relationship reciprocity
    - New insight: balance between connection reciprocity connection distance leads to network navigability!

- Other earlier attempts [Mathias&Gopal'01, Clauset&Moore'03, Sandberg&Clarke'06, Chaintreau et al.'08, Hu et al.'11]
  - Use node or link dynamics, mostly by simulation, some theoretical results on approximate settings for the navigability, none connects to reciprocity

Game Theoretic Model

- Players: n^k nodes in a k-dimensional grid
- Strategies: connection preference $r_u \in [0, +\infty)$ of node u
  - u has a long-range link to v with probability $\Pr(u \to v) \propto \frac{1}{d(u,v)^{r_u}}$
  - Indicate the preference of u in connecting to local or remote nodes
  - For convenience, we discretize $r_u \in \{0, \gamma, 2\gamma, 3\gamma, \ldots\}$, γ --- granularity

Payoff function: Distance-Reciprocity Tradeoff

- First attempt: average routing distance as payoff
  - Random network ($r_u \equiv \mathbf{0}$) seems to be the equilibrium
- Novel payoff function: distance reciprocity tradeoff

$$\pi_u(r_u, \mathbf{r}_{-u}) = \underbrace{\left(\sum_{v \neq u} p_u(v, r_u) d(u, v)\right)} \times \underbrace{\left(\sum_{v \neq u} p_u(v, r_u) p_v(u, r_v)\right)}$$

Average grid distance of a long-range link --- prefer faraway nodes to get diverse information

Average probability that the long-range link is reciprocated --- prefer mutual relationship

Theoretical Analysis

Uniform Nash Equilibria

- Theorem 1. For sufficiently large $\boldsymbol{n}$, If everyone else plays the same strategy ($\mathbf{r}_{-u} \equiv s$), the best response of $\boldsymbol{u}$ is

$$B_u(\mathbf{r}_{-u} \equiv s) = \begin{cases} k & \text{if } s > 0 \\ 0 & \text{if } s = 0 \end{cases}$$

- Corollary 2. There are only two uniform Nash equilibria:
  – Navigable small-world network ($\mathbf{r} \equiv k$)
  – Random small-world network ($\mathbf{r} \equiv \mathbf{0}$)

Intuition (Proof Sketch)

- $\mathbf{r}_{-u} \equiv s$, $s > 0$: everyone else (slightly) prefers local nodes
  - If $r_u < k$, u's long-rang links achieve good distance but poor reciprocity
  - If $r_u > k$, u's long-rang links achieve good reciprocity but poor distance
  - If $r_u = k$, u's long-rang links achieve best balance between distance and reciprocity
- $\mathbf{r}_{-u} \equiv s$, $s = 0$: everyone else connects uniformly to other nodes
  - Reciprocity is a constant regardless of r_u
  - Thus, set $r_u = 0$ to achieve the largest average grid distance

Stability of Navigable Small World --- Collusion Toleration

- What if a group of players (instead of one player) want to collude and deviate together for better payoff?

- Theorem 3. Navigable small-world network ($\mathbf{r} \equiv \boldsymbol{k}$) is a strong Nash equilibrium for sufficiently large $\boldsymbol{n}$.
  - Strong Nash: no collusion group of any size could successfully deviate from the equilibrium without someone in the group got hurt in payoff.
  - Reason: In any strategy profile, if $r_u \neq k$, u's payoff is strictly worse than its payoff in the navigable small world.

Stability of Navigable Small World ---
Random Perturbation Toleration

- What if perturbations occur at random players, without increasing payoff constraint?

- Theorem 4. In the navigable small-world network ($\mathbf{r} \equiv \boldsymbol{k}$), even if every node has an independent probability of $1 - n^{-\varepsilon}$ (for small $\boldsymbol{\varepsilon} > \mathbf{0}$) to be perturbed to an arbitrary strategy, with high probability every player $\boldsymbol{u}$ wants to set $\boldsymbol{r_u} = \boldsymbol{k}$ as its best strategy after the perturbation.
- Intuition: a small portion of randomly distributed nodes holding $\boldsymbol{r_u} = \boldsymbol{k}$ is enough to pull everyone to $\boldsymbol{r_u} = \boldsymbol{k}$.

Instability of Non-navigable Equilibria --- Not Tolerating Collusions from a Small Group

- Does any other Nash equilibrium tolerate Collusion? --- NO!
- Theorem 5. No other equilibrium tolerates the collusion of $2n^{-\varepsilon}$ (for small $\varepsilon > 0$) fraction of players.
- Intuition: Dual aspect of Theorem 4 --- a small portion of evenly distributed nodes collude and set $r_u = k$ is enough to pull everyone to $r_u = k$.

Instability of Non-navigable Equilibria --- Not Tolerating Random Perturbations

- Does any other Nash equilibrium tolerate random perturbation?
  - No if perturbed players could set strategy to k (by Theorem 4)
- What if the target strategy set after perturbation does not contain k?

- Theorem 6. In the random small world ($\mathbf{r} \equiv \mathbf{0}$), for an arbitrary finite target strategy set S after the perturbation ($\beta = \max S > 0$), if every u is perturbed to every $s \in S \setminus \{0\}$ with probability at least $n^{-\frac{(k-1)\varepsilon}{k+\beta}}$ (for small $\varepsilon > 0$), then with high probability the best strategy for every u after the perturbation is $r_u = k$.

Implication of Theoretical Analysis

- Navigable small world is the only stable state of the system
  - Once in it, any size of collusion, or large random perturbation cannot shake the system out of navigable small world
  - If the system temporarily gets stuck at other states (other equilibria)
    - Small size collusion can bring the system back to navigable small world
    - Small size random perturbation can also bring the system back to navigable small world

Empirical Evaluation

Grid size: 100 x 100

Stability of NE under Perturbation

- At time step 0, each player is perturbed independently with probability p.
- At time step $t > 0$, every player picks the best strategy based on the strategies of others in the previous step.

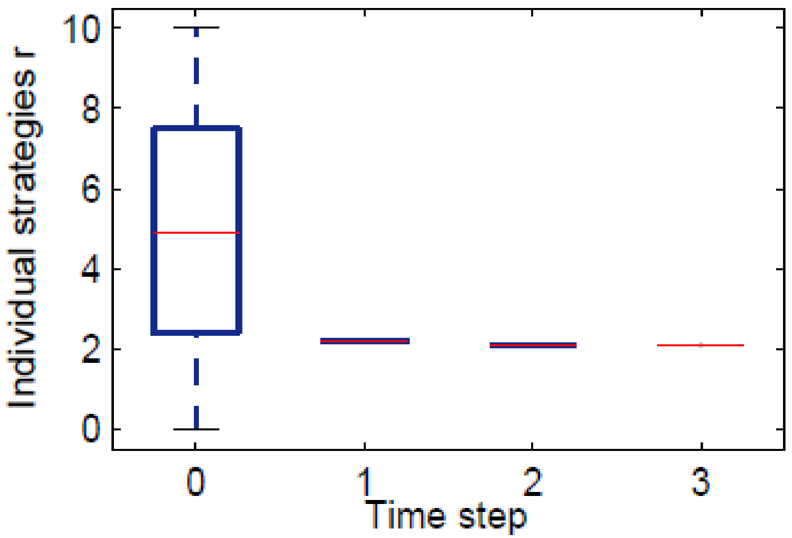


Figure 6: The return to navigable small-world NE (perturbed probability p=1).

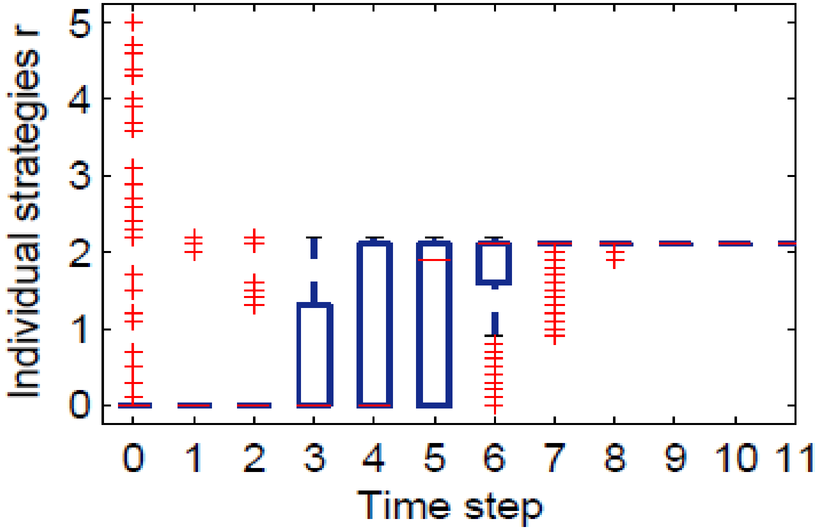
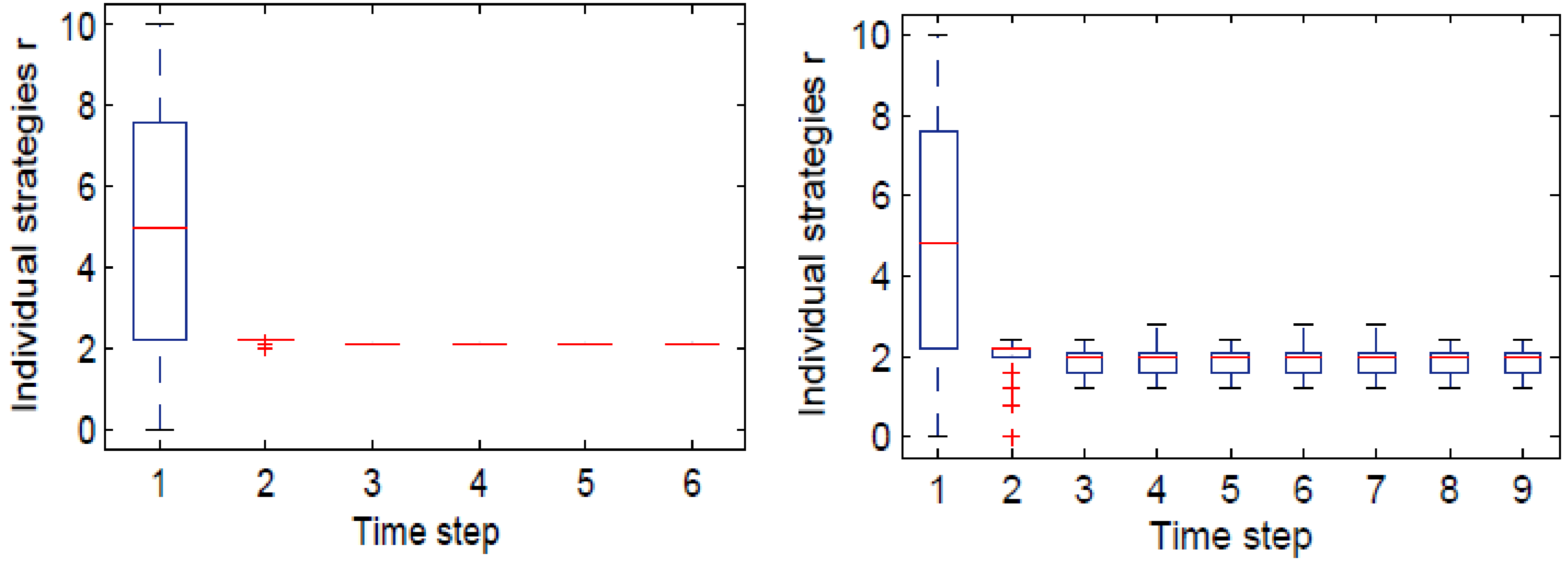


Figure 7: From random NE to small-world NE (perturbed probability p=0.01).

DRB Game with Limited Knowledge (1)

- **Scenario 1: knowing friends' strategies.**
- At every step $t \geq 0$, each player u creates q out-going long-range links based on her current strategy, and learns the (noisy) connection preferences of these q long-range contacts, then infer others' connection preferences.



(a) Noise $\varepsilon = 0$. (b) Noise $\varepsilon \sim N(0, 0.5)$

Figure 8: Network evolution where each player only knows the strategies of their friends.

DRB Game with Limited Knowledge (2)

- **Scenario 2: No information about others' strategies.**
- a player creates a certain number of links with the current strategy, and computes the payoff by multiplying the average link distance and the percentage of reciprocal links.
- at each step each player only has one chance to slightly modify her current strategy. If the new strategy yields better payoff, the player would adopt the new strategy.

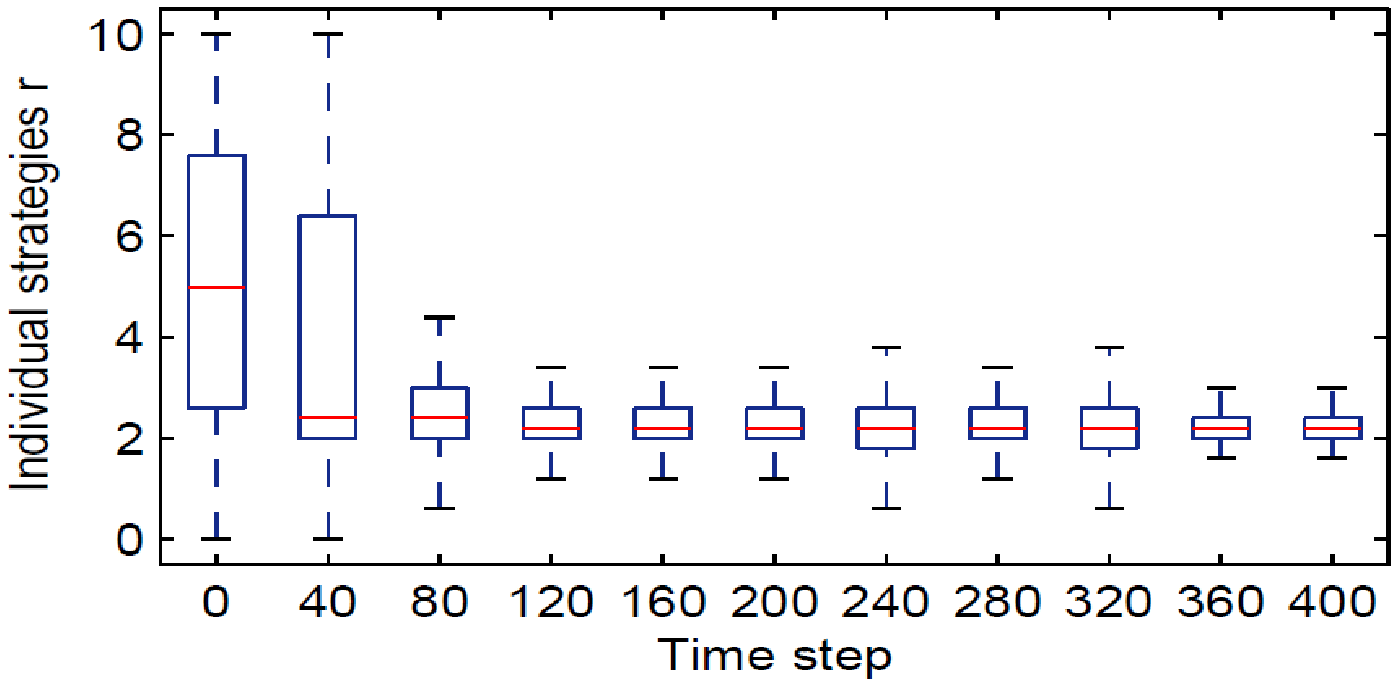


Figure 9: Network evolution where players have no knowledge of strategies of others.

Conclusion & Future Work

- The first model connecting reciprocity with navigability

$$\text{Distance x Reciprocity} \Rightarrow \text{Navigability}$$

  - Navigable small world is the only stable system state
  - Strong theoretical and empirical support

- Future work
  - Non-uniform population distribution
  - Arbitrary base graph
  - Other more general long-range link distribution than power-law
  - Integrating with node mobility and link dynamics

Thanks, and questions?