Stochastic Online Greedy Learning with Semi-bandit Feedbacks

(Full Version Including Appendices)

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Abstract

The greedy algorithm is extensively studied in the field of combinatorial optimization for decades. In this paper, we address the online learning problem when the input to the greedy algorithm is stochastic with unknown parameters that have to be learned over time. We first propose the greedy regret and ϵ-quasi greedy regret as learning metrics comparing with the performance of offline greedy algorithm. We then propose two online greedy learning algorithms with semi-bandit feedbacks, which use multi-armed bandit and pure exploration bandit policies at each level of greedy learning, one for each of the regret metrics respectively. Both algorithms achieve $O(\log T)$ problem-dependent regret bound (T being the time horizon) for a general class of combinatorial structures and reward functions that allow greedy solutions. We further show that the bound is tight in T and other problem instance parameters.

1 Introduction

The greedy algorithm is simple and easy-to-implement, and can be applied to solve a wide range of complex optimization problems, either with exact solutions (e.g. minimum spanning tree [22, 29]) or approximate solutions (e.g. maximum coverage [12] or influence maximization [19]). Moreover, for many practical problems, the greedy algorithm often serves as the first heuristic of choice and performs well in practice even when it does not provide a theoretical guarantee.

The classical greedy algorithm assumes that a certain *reward function* is given, and it constructs the solution iteratively. In *each phase*, it searches for a local optimal element to maximize the marginal gain of reward, and add it to the solution. We refer to this case as the *offline greedy* algorithm with a given reward function, and the corresponding problem the *offline problems*. The phase-by-phase process of the greedy algorithm naturally forms a *decision sequence* to illustrate the decision flow in finding the solution, which is named as the *greedy sequence*. We characterize the decision class as an *accessible set system*, a general combinatorial structure encompassing many interesting problems.

In many real applications, however, the reward function is *stochastic* and is not known in advance, and the reward is only instantiated based on the unknown distribution after the greedy sequence is selected. For example, in the influence maximization problem [19], social influence are propagated in a social network from the selected seed nodes following a stochastic model with unknown parameters, and one wants to find the optimal seed set of size k that generates the largest influence spread, which is the expected number of nodes influenced in a cascade. In this case, the reward of seed selection is only instantiated after the seed selection, and is only one of the random outcomes. Therefore, when the stochastic reward function is unknown, we aim at maximizing the expected reward overtime while gradually learning the key parameters of the expected reward functions. This falls in the domain of *online learning*, and we refer the *online algorithm* as the strategy of the *player*, who makes sequential decisions, interacts with the environment, obtains feedbacks, and accumulates her reward. For *online greedy algorithms* in particular, at each time step the player selects and plays a candidate decision sequence while the environment instantiates the reward function, and then the player collects the values of instantiated function at every phase of the decision sequence as the feedbacks (thus the name of *semi-bandit feedbacks* [2]), and takes the value of the final phase as the reward cumulated in this step.

The typical objective for an online algorithm is to make sequential decisions against the optimal solution in the offline problem where the reward function is known a priori. For online greedy algorithms, instead, we compare it with the solution of the offline greedy algorithm, and minimize their gap of the cumulative reward over time, termed as the *greedy regret*. Furthermore, in some problems such as influence maximization, the reward function is estimated with error even for the offline problem [19] and thus the greedily selected element at each phase may contain some ϵ error. We call such greedy sequence as *ϵ-quasi greedy sequence*. To accommodate these cases, we also define the metric of *ϵ-quasi greedy regret*, which compares the online solution against the minimum offline solution from all ϵ-quasi greedy sequences.

In this paper, we propose two online greedy algorithms targeted at two regret metrics respectively. The first algorithm OG-UCB uses the stochastic multi-armed bandit (MAB) [25, 8], in particular the well-known UCB policy [3] as the building block to minimize the greedy regret. We apply the UCB policy to every phase by associating the confidence bound to each arm, and then choose the arm having the highest upper confidence bound greedily in the process of decision. For the second scenario where we allow tolerating ϵ-error for each phase, we propose a *first-explore-then-exploit* algorithm OG-LUCB to minimize the ϵ-quasi greedy regret. For every phase in the greedy process, OG-LUCB applies the LUCB policy [18, 9] which depends on the upper and lower confidence bound to eliminate arms. It first explores each arm until the lower bound of one arm is higher than the upper bound of any other arm within an ϵ-error, then the stage of current phase is switched to exploit that best arm, and continues to the next phase. Both OG-UCB and OG-LUCB achieve the problem-dependent $O(\log T)$ bound in terms of the respective regret metrics, where the coefficients in front of T depends on direct elements along the greedy sequence (a.k.a., its *decision frontier*) corresponding to the instance of learning problem. The two algorithms have complementary advantages: when we really target at greedy regret (setting ϵ to 0 for OG-LUCB), OG-UCB has a slightly better regret guarantee and does not need an artificial switch between exploration and exploitation; when we are satisfied with ϵ-quasi greedy regret, OG-LUCB works but OG-UCB cannot be adapted for this case and may suffer a larger regret. We also show a problem instance in this paper, where the upper bound is tight to the lower bound in T and other problem parameters.

We further show our algorithms can be easily extended to the knapsack problem, and applied to the stochastic online maximization for consistent functions and submodular functions, etc., in the supplementary material.

To summarize, our contributions include the following: (a) To the best of our knowledge, we are the first to propose the framework using the greedy regret and ϵ-quasi greedy regret to characterize the online performance of the stochastic greedy algorithm for different scenarios, and it works for a wide class of accessible set systems and general reward functions; (b) We propose Algorithms OG-UCB and OG-LUCB that achieve the problem-dependent $O(\log T)$ regret bound; and (c) We also show that the upper bound matches with the lower bound (up to a constant factor).

Due to the space constraint, the analysis of algorithms, applications and empirical evaluation of the lower bound are moved to the supplementary material.

Related Work. The multi-armed bandit (MAB) problem for both stochastic and adversarial settings [25, 4, 6] has been widely studied for decades. Most work focus on minimizing the cumulative regret over time [3, 16], or identifying the optimal solution in terms of pure exploration bandits [1, 18, 7]. Among those work, there is one line of research that generalizes MAB to combinatorial learning problems [8, 15, 2, 11, 24, 26, 9]. Our paper belongs to this line considering stochastic learning with semi-bandit feedbacks, while we focus on the greedy algorithm, the structure and its performance measure, which have not been addressed.

The classical greedy algorithms in the offline setting are studied in many applications [22, 29, 12, 5], and there is a line of work [17, 20] focusing on characterizing the greedy structure for solutions. We adopt their characterizations of accessible set systems to the online setting of the greedy learning. There is also a branch of work using the greedy algorithm to solve online learning problem, while they require the knowledge of the exact form of reward function, restricting to special functions such as linear [2, 23] and submodular rewards [31, 14]. Our work does not assume the exact form, and it covers a much larger class of combinatorial structures and reward functions.

2 Preliminaries

Online combinatorial learning problem can be formulated as a repeated game between *the environment* and *the player* under stochastic multi-armed bandit framework.

Let $E = \{e_1, e_2, \dots, e_n\}$ be a finite *ground set* of size n, and $\mathcal{F}$ be a collection of subsets of E. We consider the *accessible set system* $(E, \mathcal{F})$ satisfying the following two axioms: (1) $\emptyset \in \mathcal{F}$; (2) If $S \in \mathcal{F}$ and $S \neq \emptyset$, then there exists some e in E, s.t., $S \setminus \{e\} \in \mathcal{F}$. We define any set $S \subseteq E$ as a *feasible set* if $S \in \mathcal{F}$. For any $S \in \mathcal{F}$, its *accessible set* is defined as $\mathcal{N}(S) := \{e \in E \setminus S : S \cup \{e\} \in \mathcal{F}\}$. We say feasible set S is *maximal* if $\mathcal{N}(S) = \emptyset$. Define the *largest length* of any feasible set as $m := \max_{S \in \mathcal{F}} |S|$ ($m \leq n$), and the *largest width* of any feasible set as $W := \max_{S \in \mathcal{F}} |\mathcal{N}(S)|$ ($W \leq n$). We say that such an *accessible set system* $(E, \mathcal{F})$ is the *decision class* of the player. In the class of combinatorial learning problems, the size of $\mathcal{F}$ is usually very large (e.g., exponential in m, W and n).

Beginning with an empty set, the *accessible set system* $(E, \mathcal{F})$ ensures that any feasible set S can be acquired by adding elements one by one in some order (cf. Lemma A.1 in the supplementary material for more details), which naturally forms the decision process of the player. For convenience, we say the player can choose a *decision sequence*, defined as an ordered feasible sets $\sigma := \langle S_0, S_1, \dots, S_k \rangle \in \mathcal{F}^{k+1}$ satisfying that $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k$ and for any $i = 1, 2, \dots, k$, $S_i = S_{i-1} \cup \{s_i\}$ where $s_i \in \mathcal{N}(S_{i-1})$. Besides, define decision sequence σ as *maximal* if and only if S_k is maximal.

Let Ω be an arbitrary set. The environment draws i.i.d. samples from Ω as $\omega_1, \omega_2, \dots$, at *each time* $t = 1, 2, \dots$, by following a predetermined but unknown distribution. Consider reward function $f : \mathcal{F} \times \Omega \to \mathbb{R}$ that is bounded, and it is non-decreasing[1] in the first parameter, while the exact form of function is agnostic to the player. We use a shorthand $f_t(S) := f(S, \omega_t)$ to denote the reward for any given S at time t, and denote the expected reward as $\overline{f}(S) := \mathbb{E}_{\omega_1}[f_1(S)]$, where the expectation $\mathbb{E}_{\omega_t}$ is taken from the randomness of the environment at time t. For ease of presentation, we assume that the reward function for any time t is normalized with arbitrary alignment as follows: (1) $f_t(\emptyset) = L$ (for any constant $L \geq 0$); (2) for any $S \in \mathcal{F}, e \in \mathcal{N}(S)$, $f_t(S \cup \{e\}) - f_t(S) \in [0, 1]$. Therefore, reward function $f(\cdot, \cdot)$ is implicitly bounded within $[L, L + m]$.

We extend the concept of arms in MAB, and introduce notation $a := e|S$ to define *an arm*, representing the selected element e based on the prefix S, where S is a feasible set and $e \in \mathcal{N}(S)$; and define $\mathcal{A} := \{e|S : \forall S \in \mathcal{F}, \forall e \in \mathcal{N}(S)\}$ as the *arm space*. Then, we can define the *marginal reward* for function f_t as $f_t(e|S) := f_t(S \cup \{e\}) - f_t(S)$, and the *expected marginal reward* for $\overline{f}$ as $\overline{f}(e|S) := \overline{f}(S \cup \{e\}) - \overline{f}(S)$. Notice that the use of arms characterizes the marginal reward, and also indicates that it is related to the player's previous decision.

2.1 The Offline Problem and The Offline Greedy Algorithm

In the *offline problem*, we assume that $\overline{f}$ is provided as a value oracle. Therefore, the objective is to find the optimal solution $S^* = \arg\max_{S \in \mathcal{F}} \overline{f}(S)$, which only depends on the player's decision. When the optimal solution is computationally hard to obtain, usually we are interested in finding a feasible set $S^+ \in \mathcal{F}$ such that $\overline{f}(S^+) \geq \alpha \overline{f}(S^*)$ where $\alpha \in (0, 1]$, then S^+ is called an α-*approximation* solution. That is a typical case where the greedy algorithm comes into play.

The *offline greedy* algorithm is a local search algorithm that refines the solution *phase by phase*. It goes as follows: (a) Let $G_0 = \emptyset$; (b) For each phase $k = 0, 1, \dots$, find $g_{k+1} = \arg\max_{e \in \mathcal{N}(G_k)} \overline{f}(e|G_k)$, and let $G_{k+1} = G_k \cup \{g_{k+1}\}$; (c) The above process ends when $\mathcal{N}(G_{k+1}) = \emptyset$ (G_{k+1} is maximal). We define the maximal decision sequence $\sigma^{\mathsf{G}} :=$

[1]Therefore, the optimal solution is a maximal decision sequence.

$\langle G_0, G_1, \ldots, G_{m^{\mathsf{G}}} \rangle$ (m^{G} is its length) found by the offline greedy as the *greedy sequence*. For simplicity, we assume that it is unique.

One important feature is that the greedy algorithm uses a polynomial number of calls ($\mathrm{poly}(m, W, n)$) to the offline oracle, even though the size of $\mathcal{F}$ or $\mathcal{A}$ may be exponentially large.

In some cases such as the offline influence maximization problem [19], the value of $\overline{f}(\cdot)$ can only be accessed with some error or estimated approximately. Sometimes, even though $\overline{f}(\cdot)$ can be computed exactly, we may only need an approximate maximizer in each greedy phase in favor of computational efficiency (e.g., efficient submodular maximization [27]). To capture such scenarios, we say a maximal decision sequence $\sigma = \langle S_0, S_1, \ldots, S_{m'} \rangle$ is an ϵ-*quasi greedy sequence* ($\epsilon \geq 0$), if the greedy decision can tolerate ϵ error every phase, i.e., for each $k = 0, 1, \ldots, m'-1$ and $S_{k+1} = S_k \cup \{s_{k+1}\}$, $\overline{f}(s_{k+1}|S_k) \geq \max_{s \in \mathcal{N}(S_k)} \overline{f}(s|S_k) - \epsilon$. Notice that there could be many ϵ-quasi greedy sequences, and we denote $\sigma^{\mathsf{Q}} := \langle Q_0, Q_1, \ldots, Q_{m^{\mathsf{Q}}} \rangle$ (m^{Q} is its length) as the one with the minimum reward, that is $\overline{f}(Q_{m^{\mathsf{Q}}})$ is minimized over all ϵ-quasi greedy sequences.

2.2 The Online Problem

In the online case, in constrast $\overline{f}$ is not provided. The player can only access one of functions $f_1, f_2, \ldots$, generated by the environment, for each time step during a repeated game.

For each time t, the game proceeds in the following three steps: (1) The environment draws i.i.d. sample $\omega_t \in \Omega$ from its predetermined distribution without revealing it; (2) the player may, based on her previous knowledge, select a decision sequence $\sigma^t = \langle S_0, S_1, \ldots, S_{m^t} \rangle$, which reflects the process of her decision phase by phase; (3) then, the player plays σ^t and gains reward $f_t(S_{m^t})$, while observes intermediate feedbacks $f_t(S_0), f_t(S_1), \ldots, f_t(S_{m^t})$ to update her knowledge. We refer such feedbacks as *semi-bandit feedbacks* in the decision order.

For any time $t = 1, 2, \ldots$, denote $\sigma^t = \langle S_0^t, S_1^t, \ldots, S_{m^t}^t \rangle$ and $S^t := S_{m^t}^t$. The player is to make sequential decisions, and the classical objective is to minimize the cumulative gap of rewards against the optimal solution [3] or the approximation solution [11]. For example, when the optimal solution $S^* = \arg\max_{S \in \mathcal{F}} \mathbb{E}[f_1(S)]$ can be solved in the offline problem, we minimize the expected *cumulative regret* $R(T) := T \cdot \mathbb{E}[f_1(S^*)] - \sum_{t=1}^{T} \mathbb{E}[f_t(S^t)]$ over the time horizon T, where the expectation is taken from the randomness of the environment and the possible random algorithm of the player. In this paper, we are interested in online algorithms that are comparable to the solution of the offline greedy algorithm, namely the greedy sequence $\sigma^{\mathsf{G}} = \langle G_0, G_1, \ldots, G_{m^{\mathsf{G}}} \rangle$. Thus, the objective is to minimize the *greedy regret* defined as

$$R^{\mathsf{G}}(T) := T \cdot \mathbb{E}[f_1(G_{m^{\mathsf{G}}})] - \sum_{t=1}^{T} \mathbb{E}\left[f_t(S^t)\right]. \tag{1}$$

Given $\epsilon \geq 0$, we define the ϵ-*quasi greedy regret* as

$$R^{\mathsf{Q}}(T) := T \cdot \mathbb{E}[f_1(Q_{m^{\mathsf{Q}}})] - \sum_{t=1}^{T} \mathbb{E}\left[f_t(S^t)\right], \tag{2}$$

where $\sigma^{\mathsf{Q}} = \langle Q_0, Q_1, \ldots, Q_{m^{\mathsf{Q}}} \rangle$ is the minimum ϵ-quasi greedy sequence.

We remark that if the offline greedy algorithm provides an α-approximation solution (with $0 < \alpha \leq 1$), then the greedy regret (or ϵ-quasi greedy regret) also provides α-approximation regret, which is the regret comparing to the α fraction of the optimal solution, as defined in [11].

In the rest of the paper, our goal is to design the player's policy that is comparable to the offline greedy, in other words, $R^{\mathsf{G}}(T)/T = \overline{f}(G_{m^{\mathsf{G}}}) - \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[f_t(S^t)] = o(1)$. Thus, to achieve sublinear greedy regret $R^{\mathsf{G}}(T) = o(T)$ is our main focus.

3 The Online Greedy and Algorithm OG-UCB

In this section, we propose our Online Greedy (OG) algorithm with the UCB policy to minimize the greedy regret (defined in (1)).

Algorithm 1 OG

Require: MaxOracle

1: **for** $t = 1, 2, \ldots$ **do**

2: $S_0 \leftarrow \emptyset$; $k \leftarrow 0$; $h_0 \leftarrow$ true

3: **repeat** ▷ online greedy procedure

4: $A \leftarrow \{e|S_k : \forall e \in \mathcal{N}(S_k)\}$; $t' \leftarrow \sum_{a \in A} N(a) + 1$

5: $(s_{k+1}|S_k, h_k) \leftarrow \mathrm{MaxOracle}\left(A, \hat{X}(\cdot), N(\cdot), t'\right)$ ▷ find the current maximal

6: $S_{k+1} \leftarrow S_k \cup \{s_{k+1}\}$; $k \leftarrow k + 1$

7: **until** $\mathcal{N}(S_k) = \emptyset$ ▷ until a maximal sequence is found

8: Play sequence $\sigma^t \leftarrow \langle S_0, \ldots, S_k \rangle$, observe $\{f_t(S_0), \ldots, f_t(S_k)\}$, and gain $f_t(S_k)$.

9: **for all** $i = 1, 2, \ldots, k$ **do** ▷ update according to signals from MaxOracle

10: **if** $h_0, h_1, \cdots, h_{i-1}$ **are all** true **then**

11: Update $\hat{X}(s_i|S_{i-1})$ and $N(s_i|S_{i-1})$ according to (3).

Subroutine 2 UCB$(A, \hat{X}(\cdot), N(\cdot), t)$ to implement MaxOracle

Setup: confidence radius $\mathrm{rad}_t(a) := \sqrt{\frac{3 \ln t}{2N(a)}}$, for each $a \in A$

if $\exists a \in A$, $\hat{X}(a)$ is not initialized **then** ▷ break ties arbitrarily

return (a, true) ▷ to initialize arms

else ▷ apply UCB's rule

$I_t^+ \leftarrow \arg\max_{a \in A} \left\{\hat{X}(a) + \mathrm{rad}_t(a)\right\}$, and **return** (I_t^+, true)

For any arm $a = e|S \in \mathcal{A}$, playing a at each time t yields the marginal reward as a random variable $X_t(a) = f_t(a)$, in which the random event $\omega_t \in \Omega$ is i.i.d., and we denote $\mu(a)$ as its true mean (i.e., $\mu(a) := \mathbb{E}[X_1(a)]$). Let $\hat{X}(a)$ be the empirical mean for the marginal reward of a, and $N(a)$ be the counter of the plays. More specifically, denote $\hat{X}_t(a)$ and $N_t(a)$ for particular $\hat{X}(a)$ and $N(a)$ at the beginning of the time step t, and they are evaluated as follows:

$$\hat{X}_t(a) = \frac{\sum_{i=1}^{t-1} f_i(a)\mathbb{I}_i(a)}{\sum_{i=1}^{t-1} \mathbb{I}_i(a)}, \quad N_t(a) = \sum_{i=1}^{t-1} \mathbb{I}_i(a), \tag{3}$$

where $\mathbb{I}_i(a) \in \{0, 1\}$ indicates whether a is updated at time i. In particular, assume that our algorithm is lazy-initialized so that each $\hat{X}(a)$ and $N(a)$ is 0 by default, until a is played.

The *Online Greedy* algorithm (OG) proposed in Algorithm 1 serves as a meta-algorithm allowing different implementations of Subroutine MaxOracle. For every time t, OG calls MaxOracle (Line 5, to be specified later) to find the local maximal phase by phase, until the decision sequence σ^t is made. Then, it plays sequence σ^t, observes feedbacks and gains the reward (Line 8). Meanwhile, OG collects the Boolean signals (h_k) from MaxOracle during the greedy process (Line 5), and update estimators $\hat{X}(\cdot)$ and $N(\cdot)$ according to those signals (Line 10). On the other hand, MaxOracle takes accessible arms A, estimators $\hat{X}(\cdot), N(\cdot)$, and counted time t', and returns an arm from A and signal $h_k \in \{\text{true}, \text{false}\}$ to instruct OG whether to update estimators for the following phase.

The classical UCB [3] can be used to implement MaxOracle, which is described in Subroutine 2. We term our algorithm OG, in which MaxOracle is implemented by Subroutine 2 UCB, as Algorithm OG-UCB. A few remarks are in order: First, Algorithm OG-UCB chooses an arm with the highest upper confidence bound for each phase. Second, the signal h_k is *always* true, meaning that OG-UCB always update empirical means of arms along the decision sequence. Third, because we use lazy-initialized $\hat{X}(\cdot)$ and $N(\cdot)$, the memory is allocated only when it is needed.

3.1 Regret Bound of OG-UCB

For any feasible set S, define the *greedy element for* S as $g_S^* := \arg\max_{e \in \mathcal{N}(S)} \overline{f}(e|S)$, and we use $\mathcal{N}_-(S) := \mathcal{N}(S) \setminus \{g_S^*\}$ for convenience. Denote $\mathcal{F}^\dagger := \{S \in \mathcal{F} : S \text{ is maximal}\}$ as the collection of all maximal feasible sets in $\mathcal{F}$. We use the following gaps to measure the performance of the algorithm.

Definition 3.1 (Gaps)**.** The *gap* between the maximal greedy feasible set $G_{m^{\mathsf{G}}}$ and any $S \in \mathcal{F}$ is defined as $\Delta(S) := \overline{f}(G_{m^{\mathsf{G}}}) - \overline{f}(S)$ if it is positive, and 0 otherwise. We define the *maximum gap* as $\Delta_{\max} = \overline{f}(G_{m^{\mathsf{G}}}) - \min_{S \in \mathcal{F}^\dagger} \overline{f}(S)$, which is the worst penalty for any maximal feasible set. For any arms $a = e|S \in \mathcal{A}$, we define the *unit gap* of a (i.e., the gap for one phase) as

$$\Delta(a) = \Delta(e|S) := \begin{cases} \overline{f}(g_S^*|S) - \overline{f}(e|S), & e \neq g_S^* \\ \overline{f}(g_S^*|S) - \max_{e' \in \mathcal{N}_-(S)} \overline{f}(e'|S), & e = g_S^* \end{cases}. \tag{4}$$

For any arms $a = e|S \in \mathcal{A}$, we define the *sunk-cost gap* (irreversible once selected) as

$$\Delta^*(a) = \Delta^*(e|S) := \max\left\{ \overline{f}(G_{m^{\mathsf{G}}}) - \min_{V : V \in \mathcal{F}^\dagger, S \cup \{e\} \prec V} \overline{f}(V), 0 \right\}, \tag{5}$$

where for two feasible sets A and B, $A \prec B$ means that A is a prefix of B in some decision sequence, that is, there exists a decision sequence $\sigma = \langle S_0 = \emptyset, S_1, \dots, S_k \rangle$ such that $S_k = B$ and for some $j < k$, $S_j = A$. Thus, $\Delta^*(e|S)$ means the largest gap we may have after we have fixed our prefix selection to be $S \cup \{e\}$, and is upper bounded by $\Delta_{\max}$.

Definition 3.2 (Decision frontier)**.** For any decision sequence $\sigma = \langle S_0, S_1, \dots, S_k \rangle$, define *decision frontier* $\Gamma(\sigma) := \bigcup_{i=1}^{k} \{e|S_{i-1} : e \in \mathcal{N}(S_{i-1})\} \subseteq \mathcal{A}$ as the arms need to be explored in the decision sequence σ, and $\Gamma_-(\sigma) := \bigcup_{i=1}^{k} \{e|S_{i-1} : \forall e \in \mathcal{N}_-(S_{i-1})\}$ similarly.

Theorem 3.1 (Greedy regret bound)**.** *For any time T, Algorithm* OG-UCB *(Algorithm 1 with Subroutine 2) can achieve the greedy regret*

$$R^{\mathsf{G}}(T) \leq \sum_{a \in \Gamma_-(\sigma^{\mathsf{G}})} \left(\frac{6\Delta^*(a) \cdot \ln T}{\Delta(a)^2} + \left(\frac{\pi^2}{3} + 1 \right) \Delta^*(a) \right), \tag{6}$$

where σ^{G} is the greedy decision sequence.

When $m = 1$, the above theorem immediately recovers the regret bound of the classical UCB [3] (with $\Delta^*(a) = \Delta(a)$). The greedy regret is bounded by $O\left(\frac{mW\Delta_{\max} \log T}{\Delta^2}\right)$ where Δ is the minimum unit gap ($\Delta = \min_{a \in \mathcal{A}} \Delta(a)$), and the memory cost is at most proportional to the regret. For a special class of linear bandits, a simple extension where we treat arms $e|S$ and $e|S'$ as the same can make OG-UCB essentially the same as OMM in [23], while the regret is $O(\frac{n}{\Delta} \log T)$ and the memory cost is $O(n)$ (cf. Appendix F.1 of the supplementary material).

4 Relaxing the Greedy Sequence with ϵ-Error Tolerance

In this section, we propose an online algorithm called OG-LUCB, which learns an ϵ-quasi greedy sequence, with the goal of minimizing the ϵ-quasi greedy regret (in (2)). We learn ϵ-quasi-greedy sequences by a *first-explore-then-exploit policy*, which utilizes results from PAC learning with a fixed confidence setting. In Section 4.1, we implement $\mathrm{MaxOracle}$ via the LUCB policy, and derive its exploration time; we then assume the knowledge of time horizon T in Section 4.2, and analyze the ϵ-quasi greedy regret; and in Section 4.3, we show that the assumption of knowing T can be further removed.

4.1 OG with a first-explore-then-exploit policy

Given $\epsilon \geq 0$ and failure probability $\delta \in (0, 1)$, we use Subroutine 3 $\mathrm{LUCB}_{\epsilon,\delta}$ to implement the subroutine $\mathrm{MaxOracle}$ in Algorithm OG. We call the resulting algorithm $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$. Specifically, Subroutine 3 is adapted from CLUCB-PAC in [9], and specialized to explore the top-one element in the support of $[0, 1]$ (i.e., set $R = \frac{1}{2}$, $\mathrm{width}(\mathcal{M}) = 2$ and $\mathrm{Oracle} = \arg\max$ in [9]). Assume that $I^{\mathsf{exploit}}(\cdot)$ is lazy-initialized. For each greedy phase, the algorithm first explores each arm in A in the *exploration* stage, during which the return flag (the second return field) is always false; when the optimal one is found (initialize $I^{\mathsf{exploit}}(A)$ with $\hat{I}_t$), it sticks to $I^{\mathsf{exploit}}(A)$ in the *exploitation* stage

Subroutine 3 $\mathrm{LUCB}_{\epsilon,\delta}(A, \hat{X}(\cdot), N(\cdot), t)$ to implement MaxOracle

Setup: $\mathrm{rad}_t(a) := \sqrt{\frac{\ln(4Wt^3/\delta)}{2N(a)}}$, for each $a \in A$; $I^{\mathsf{exploit}}(\cdot)$ to cache arms for exploitation;

if $I^{\mathsf{exploit}}(A)$ is initialized **then return** $(I^{\mathsf{exploit}}(A), \mathsf{true})$ ▷ in the *exploitation* stage
if $\exists a \in A$, $\hat{X}(a)$ is not initialized **then** ▷ break ties arbitrarily
  return (a, false) ▷ to initialize arms
else
  $\hat{I}_t \leftarrow \arg\max_{a \in A} \hat{X}(a)$
  $\forall a \in A, X'(a) \leftarrow \begin{cases} \hat{X}(a) + \mathrm{rad}_t(a), & a \neq \hat{I}_t \\ \hat{X}(a) - \mathrm{rad}_t(a), & a = \hat{I}_t \end{cases}$ ▷ perturb arms
  $I'_t \leftarrow \arg\max_{a \in A} X'(a)$
  if $X'(I'_t) - X'(\hat{I}_t) > \epsilon$ **then** ▷ not separated
    $I''_t \leftarrow \arg\max_{i \in \{\hat{I}_t, I'_t\}} \mathrm{rad}_t(i)$, and **return** (I''_t, false) ▷ in the *exploration* stage
  else ▷ separated
    $I^{\mathsf{exploit}}(A) \leftarrow \hat{I}_t$ ▷ initialize $I^{\mathsf{exploit}}(A)$ with $\hat{I}_t$
    return $(I^{\mathsf{exploit}}(A), \mathsf{true})$ ▷ in the *exploitation* stage

for the subsequent time steps, and return flag for this phase becomes true. The main algorithm OG then uses these flags in such a way that it updates arm estimates for phase i if any only if all phases for $j < i$ are already in the exploitation stage. This avoids maintaining useless arm estimates and is a major memory saving comparing to $\mathsf{OG\text{-}UCB}$.

In Algorithm $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$, we define the *total exploration time* $T^{\mathsf{E}} = T^{\mathsf{E}}(\delta)$, such that for any time $t \geq T^{\mathsf{E}}$, $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ is in the exploitation stage for all greedy phases encountered in the algorithm. This also means that after time T^{E}, in every step we play the same maximal decision sequence $\sigma = \langle S_0, S_1, \cdots, S_k \rangle \in \mathcal{F}^{k+1}$, which we call a *stable sequence*. Following a common practice, we define the *hardness coefficient* with prefix $S \in \mathcal{F}$ as

$$H_S^{\epsilon} := \sum_{e \in \mathcal{N}(S)} \frac{1}{\max\left\{\Delta(e|S)^2, \epsilon^2\right\}}, \text{ where } \Delta(e|S) \text{ is defined in (4).} \tag{7}$$

Rewrite definitions with respect to the ϵ-quasi regret. Recall that $\sigma^{\mathsf{Q}} = \langle Q_0, Q_1, \ldots, Q_{m^{\mathsf{Q}}} \rangle$ is the minimum ϵ-quasi greedy sequence. In this section, we rewrite the gap $\Delta(S) := \max\{\overline{f}(Q_{m^{\mathsf{Q}}}) - \overline{f}(S), 0\}$ for any $S \in \mathcal{F}$, the maximum gap $\Delta_{\max} := \overline{f}(Q_{m^{\mathsf{Q}}}) - \min_{S \in \mathcal{F}^{\dagger}} \overline{f}(S)$, and $\Delta^*(a) = \Delta^*(e|S) := \max\left\{\overline{f}(Q_{m^{\mathsf{Q}}}) - \min_{V: V \in \mathcal{F}^{\dagger}, S \cup \{e\} \prec V} \overline{f}(V), 0\right\}$, for any arm $a = e|S \in \mathcal{A}$.

The following theorem shows that, with high probability, we can find a stable ϵ-quasi greedy sequence, and the total exploration time is bounded.

Theorem 4.1 (High probability exploration time)**.** *Given any $\epsilon \geq 0$ and $\delta \in (0, 1)$, suppose after the total exploration time $T^{\mathsf{E}} = T^{\mathsf{E}}(\delta)$, Algorithm $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ (Algorithm 1 with Subroutine 3) sticks to a stable sequence $\sigma = \langle S_0, S_1, \cdots, S_{m'} \rangle$ where m' is its length. With probability at least $1 - m\delta$, the following claims hold: (1) σ is an ϵ-quasi greedy sequence; (2) The total exploration time satisfies that $T^{\mathsf{E}} \leq 127 \sum_{k=0}^{m'-1} H_S^{\epsilon} \ln\left(1996 W H_S^{\epsilon}/\delta\right)$,*

4.2 Time Horizon T is Known

Knowing time horizon T, we may let $\delta = \frac{1}{T}$ in $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ to derive the ϵ-quasi regret as follows.

Theorem 4.2. *Given any $\epsilon \geq 0$. When total time T is known, let Algorithm $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ run with $\delta = \frac{1}{T}$. Suppose $\sigma = \langle S_0, S_1, \cdots, S_{m'} \rangle$ is the sequence selected at time T. Define function $R^{\mathsf{Q},\sigma}(T) := \sum_{e|S \in \Gamma(\sigma)} \Delta^*(e|S) \min\left\{\frac{127}{\Delta(e|S)^2}, \frac{113}{\epsilon^2}\right\} \ln\left(1996 W H_S^{\epsilon} T\right) + \Delta_{\max} m$, where m is the largest length of a feasible set and H_S^{ϵ} is defined in (7). Then, the ϵ-quasi regret satisfies that $R^{\mathsf{Q}}(T) \leq R^{\mathsf{Q},\sigma}(T) = O\left(\frac{Wm\Delta_{\max}}{\max\{\Delta^2, \epsilon^2\}} \log T\right)$, where Δ is the minimum unit gap.*

In general, the two bounds (Theorem 3.1 and Theorem 4.2) are for different regret metrics, thus can not be directly compared. When $\epsilon = 0$, $\mathsf{OG\text{-}UCB}$ is slightly better only in the constant before $\log T$.

Algorithm 4 OG-LUCB-R (i.e., OG-LUCB with Restart)

Require: ϵ

1: **for** epoch $\ell = 1, 2, \cdots$ **do**

2: Clean $\hat{X}(\cdot)$ and $N(\cdot)$ for all arms, and restart OG-LUCB$_{\epsilon,\delta}$ with $\delta = \frac{1}{\phi_\ell}$ (defined in (8)).

3: Run OG-LUCB$_{\epsilon,\delta}$ for ϕ_ℓ time steps. (exit halfway, if the time is over.)

On other hand, when we are satisfied with ϵ-quasi greedy regret, OG-LUCB$_{\epsilon,\delta}$ may work better for some large ϵ, for the bound takes the maximum (in the denominator) of the problem-dependent term $\Delta(e|S)$ and the fixed constant ϵ term, and the memory cost is only $O(mW)$.

4.3 Time Horizon T is not Known

When time horizon T is not known, we can apply the "squaring trick", and restart the algorithm for each epoch as follows. Define *the duration of epoch* ℓ as ϕ_ℓ, and its *accumulated time* as τ_ℓ, where

$$\phi_\ell := e^{2^\ell}; \quad \tau_\ell := \begin{cases} 0, & \ell = 0 \\ \sum_{s=1}^{\ell} \phi_s, & \ell \geq 1 \end{cases}. \tag{8}$$

For any time horizon T, define the *final epoch* $K = K(T)$ as the epoch where T lies in, that is $\tau_{K-1} < T \leq \tau_K$. Then, our algorithm OG-LUCB-R is proposed in Algorithm 4. The following theorem shows that the $O(\log T)$ ϵ-quasi regret still holds, with a slight blowup of the constant hidden in the big O notation (For completeness, the explicit constant before $\log T$ can be found in Theorem D.7 of the supplementary material).

Theorem 4.3. *Given any $\epsilon \geq 0$. Use ϕ_ℓ and τ_ℓ defined in (8), and function $R^{\mathsf{Q},\sigma}(T)$ defined in Theorem 4.2. In Algorithm OG-LUCB-R, suppose $\sigma^{(\ell)} = \langle S_0^{(\ell)}, S_1^{(\ell)}, \cdots, S_{m^{(\ell)}}^{(\ell)} \rangle$ is the sequence selected by the end of ℓ-th epoch of OG-LUCB$_{\epsilon,\delta}$, where $m^{(\ell)}$ is its length. For any time T, denote final epoch as $K = K(T)$ such that $\tau_{K-1} < T \leq \tau_K$, and the ϵ-quasi regret satisfies that $R^{\mathsf{Q}}(T) \leq \sum_{\ell=1}^{K} R^{\mathsf{Q},\sigma^{(\ell)}}(\phi_\ell) = O\left(\frac{Wm\Delta_{\max}}{\max\{\Delta^2,\epsilon^2\}} \log T\right)$, where Δ is the minimum unit gap.*

5 Lower Bound on the Greedy Regret

Consider a problem of selecting one element each from m bandit instances, and the player sequentially collects prize at every phase. For simplicity, we call it the *prize-collecting problem*, which is defined as follows: For each bandit instance $i = 1, 2, \ldots, m$, denote set $E_i = \{e_{i,1}, e_{i,2}, \ldots, e_{i,W}\}$ of size W. The accessible set system is defined as $(E, \mathcal{F})$, where $E = \bigcup_{i=1}^m E_i$, $\mathcal{F} = \cup_{i=1}^m \mathcal{F}_i \cup \{\emptyset\}$, and $\mathcal{F}_i = \{S \subseteq E : |S| = i, \forall k : 1 \leq k \leq i, |S \cap E_k| = 1\}$. The reward function $f : \mathcal{F} \times \Omega \to [0, m]$ is non-decreasing in the first parameter, and the form of f is unknown to the player. Let minimum unit gap $\Delta := \min\left\{\overline{f}(g_S^*|S) - \overline{f}(e|S) : \forall S \in \mathcal{F}, \forall e \in \mathcal{N}_-(S)\right\} > 0$, where its value is also unknown to the player. The objective of the player is to minimize the greedy regret.

Denote the greedy sequence as $\sigma^{\mathsf{G}} = \langle G_0, G_1, \cdots, G_m \rangle$, and the greedy arms as $\mathcal{A}^{\mathsf{G}} = \{g_{G_{i-1}}^* | G_{i-1} : \forall i = 1, 2, \cdots, W\}$. We say an algorithm is *consistent*, if the sum of playing all arms $a \in \mathcal{A} \setminus \mathcal{A}^{\mathsf{G}}$ is in $o(T^\eta)$, for any $\eta > 0$, i.e., $\mathbb{E}[\sum_{a \in \mathcal{A} \setminus \mathcal{A}^{\mathsf{G}}} N_T(a)] = o(T^\eta)$.

Theorem 5.1. *For any consistent algorithm, there exists a problem instance of the prize-collecting problem, as time T tends to ∞, for any minimum unit gap $\Delta \in (0, \frac{1}{4})$, such that $\Delta^2 \geq \frac{2}{3W^{\xi m-1}}$ for some constant $\xi \in (0,1)$, the greedy regret satisfies that $R^{\mathsf{G}}(T) = \Omega\left(\frac{mW \ln T}{\Delta^2}\right)$.*

We remark that the detailed problem instance and the greedy regret can be found in Theorem E.2 of the supplementary material. Furthermore, we may also restrict the maximum gap $\Delta_{\max}$ to $\Theta(1)$, and the lower bound $R^{\mathsf{G}}(T) = \Omega(\frac{mW\Delta_{\max} \ln T}{\Delta^2})$, for any sufficiently large T. For the upper bound, OG-UCB (Theorem 3.1) gives that $R^{\mathsf{G}}(T) = O(\frac{mW\Delta_{\max}}{\Delta^2} \log T)$, Thus, our upper bound of OG-UCB matches the lower bound within a constant factor.

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Symbols in the main text	*Definition*
$(E, \mathcal{F})$	the ground set, and the collection of all feasible sets in Section 2
$\mathcal{N}(S)$	the accessible set from prefix S in Section 2
n	the size of ground set $\lvert E\rvert$ in Section 2
m	the maximal length of any feasible set in Section 2
W	the maximal width of any accessible set in Section 2
$a = e\lvert S,\ \mathcal{A}$	one arm, and the arm space in Section 2
$\sigma = \langle S_0, \cdots, S_k\rangle$	a decision sequence in Section 2
$f(\cdot,\cdot)$, $f_t(\cdot)$	the reward function, and its shorthand in Section 2
$\overline{f}(\cdot)$	the expected reward function in Section 2
$\sigma^{\mathsf{G}} = \langle G_0, \cdots, G_{m^{\mathsf{G}}}\rangle$	the greedy sequence where m^{G} is its length in Section 2.1
$\sigma^{\mathsf{Q}} = \langle Q_0, \cdots, Q_{m^{\mathsf{Q}}}\rangle$	the minimum ϵ-quasi greedy sequence where m^{Q} is its length in Section 2.1
$\Gamma(\sigma)$	the decision frontier of σ in Section 3.1
$\Gamma_-(\sigma)$	the decision frontier of σ, excluding all greedy elements in Section 3.1
$R(T)$	the cumulative regret in Section 2.2
$R^{\mathsf{G}}(T)$, $R^{\mathsf{Q}}(T)$,	the greedy regret ((1)), the ϵ-quasi greedy regret ((2))
$R^{\alpha}(T)$	the α-approximation regret discussed in Section 2.2 (formally defined in Definition B.1)
$\mathcal{F}^{\dagger}$	the collection of all maximal feasible sets in Section 3.1
$\hat{X}(a)$, $N(a)$, $X(a)$	the mean estimator of $\{f_t(a)\}_{t=1}^{\infty}$, the counter, and the true mean in Section 3
$\hat{X}_t(a)$, $N_t(a)$	the particular $\hat{X}(a)$ and $N(a)$ at the beginning of the time step t, in Section 3
g_S^*	the greedy element of prefix S in Section 3.1
$\mathcal{N}_-(S)$	the accessible set from prefix S, excluding the greedy element, in Section 3.1
$\Delta(S)$	the gap between σ^{G} (or σ^{Q}) and S in Definition 3.1 (or rewritten in Section 4.1)
$\Delta_{\max}$	the maximum gap of $\Delta(S)$ defined in Definition 3.1 (or rewritten in Section 4.1)
$\Delta(a)$ (or $\Delta(e\lvert S)$)	the unit gap of arm $a = e\lvert S$ defined in (4)
$\Delta^*(a)$ (or $\Delta^*(e\lvert S)$)	the maximum gap of selecting $a = e\lvert S$ defined in (5) (rewritten in Section 4.1)
H_S^{ϵ}	the hardness coefficient in Section 4.1
T^{E}	the total exploration time for $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ until a stable sequence is found
$N_t^{\mathsf{E}}(a)$	the counter of playing arm a during the exploration before time t in Section 4.1
ϕ_ℓ, τ_ℓ	the duration of ℓ-th epoch, and its accumulated time defined in (8)
$K = K(T)$	the final epoch of time horizon T defined in Section 4.3
$R^{\mathsf{Q},\sigma}(T)$	an intermediate form of the ϵ-quasi regret for sequence σ defined in Theorem 4.2

Table 1: List of symbols in the main text.

Appendix

The appendix is organized as follows.

Above all, for ease of reading, we list the symbols and their definitions in the main text in Table 1.

In Appendix A, we show that any feasible set in $\mathcal{F}$ is accessible by a decision sequence, and demonstrate the concrete decision classes in our model with a few examples.

In Appendix B, we formally define *α-approximation regret*, and give some simple propositions to show that such a regret can be derived immediately from the greedy regret and the ϵ-quasi greedy regret if the offline solution is an α-approximation solution.

In Appendix C, we first establish lemmas by the set decomposition for Algorithm OG-UCB, and prove that OG-UCB can achieve the problem-dependent $O(\log T)$ greedy regret.

In Appendix D, we relax the greedy sequence to tolerate ϵ-error. We prove the high-probability exploration time of for Algorithm $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ in the beginning. By letting $\delta = \frac{1}{T}$ with the known time horizon T, we show that the first-explore-then-exploit policy of Algorithm $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ (associate $\delta = 1/T$) achieves the problem-dependent $O(\log T)$ ϵ-quasi regret bound. Then, we show that Algorithm OG-LUCB-R utilizing $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ can remove the dependence on T, and the $O(\log T)$ bound holds with a slight compensation for its constant.

In Appendix E, we construct a problem instance of the prize-collecting problem, and show the upper bound is tight with the lower bound (up to a constant factor).

In Appendix F, we show a simple extension to recover $O(\frac{n}{\Delta}\log T)$ bound for a linear bandit with matroid constraints, and discuss an extension of our model to the Knapsack problem.

In Appendix G, we first apply our algorithms to the top-m selection problem with a consistent function and the online submodular problem, and show the α-approximation regret when the offline greedy yields an α-approximation solution. Then we discuss the application to the online version of the influence maximization problem and the probabilistic set cover problem, which may remove the assumption on one particular diffusion model and avoid the issue of model misspecification.

In Appendix H, we evaluate the lower bound for the prize-collecting problem, and compare the regret of OG-UCB and the lower bound numerically.

A The Accessible Set System $(E, \mathcal{F})$

An accessible set system $(E, \mathcal{F})$ (in Section 2) satisfies two axioms:

(A1: Triviality axiom). $\emptyset \in \mathcal{F}$;

(A2: Accessibility axiom). If $S \in \mathcal{F}$ and $S \neq \emptyset$, then there exists some e in E, s.t., $S \setminus \{e\} \in \mathcal{F}$.

We claim that any set $S \in \mathcal{F}$ can be obtained by adding one element at a time from the empty set. Formally, we have the following fact.

Lemma A.1 (Accessibility of any feasible set)**.** *For any set $S \in \mathcal{F}$, denote $k = |S|$. There exists a sequence $\langle S_0, S_1, \cdots, S_k \rangle$, such that: (1) $S_0, S_1, \cdots, S_k \in \mathcal{F}$; (2) $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = S$; (3) $|S_i| - |S_{i-1}| = 1$, for each $i = 1, 2, \ldots, k$.*

Proof. If $S = \emptyset$, it is obvious that the sequence $\{S_0\}$ satisfies the above three properties, therefore we only focus on a non-empty S, that is $k \geq 1$.

For any set $S \in \mathcal{F}$, let $S_k = S$. From Axiom A2, we know that there exists some $e \in E$, $S_k \setminus \{e\} \in \mathcal{F}$, thus we can denote $S_{k-1} = S_k \setminus \{e\}$. We can carry on this procedure, and apply Axiom A2 iteratively. Then we can get S_i, for each $i = k-1, k-2, \ldots, 0$. Therefore, the sequence $\langle S_0, S_1, \cdots, S_k \rangle$ we found satisfies the three properties, which ends the proof. ☐

Our characterization of accessible set system $(E, \mathcal{F})$ encompasses greedoids and matroids as special cases, which is studied in the previous literature [5, 17]. More specifically, from [17], we know that:

Definition A.1 (Greedoid)**.** A greedoid is an accessible set system $(E, \mathcal{F})$ satisfying:

(A3: Augmentation axiom). For all $S, T \in \mathcal{F}$ such that $|S| > |T|$, there is an $x \in S \setminus T$ such that $T \cup \{x\} \in \mathcal{F}$.

Definition A.2 (Matroid)**.** A matroid is a greedoid $(E, \mathcal{F})$ satisfying:

(A4: Hereditary axiom). If $S \in \mathcal{F}$ and $T \subseteq S$, then $T \in \mathcal{F}$.

In addition, we list a few concrete examples that can fit into our model.

Example A.1 (Top-m selection)**.** In accessible set system $(E, \mathcal{F})$, $\mathcal{F}$ contains all sets $S \subseteq E$, where $|S| \leq m$. Therefore, a maximal feasible set is a subset of size m.

Example A.2 (Spanning Tree)**.** Given a graph $\mathcal{G} = (V, E)$, a forest F in $\mathcal{G}$ is a subset of edges in E that does not contains a cycle. The corresponding accessible set system is $(E, \mathcal{F})$, where $\mathcal{F} = \{F \subseteq E : F \text{ is a forest}\}$. The maximal set of a spanning tree constraint is a forest that cannot include more edges.

Example A.3 (Gaussian Elimination Greedoids [32])**.** Let $M = (m_{ij}) \in \mathbb{K}^{m \times n}$ be a $n \times m$ matrix over an arbitrary field $\mathbb{K}$. The accessible set system is $(E, \mathcal{F})$ where $E = \{1, 2, \cdots, n\}$ and $\mathcal{F} = \{A \subseteq E : \text{the submatrix } M_{\{1,2,\cdots,|A|\},A} \text{ is non-singular}\}$. It corresponds to the procedure of performing Gaussian Elimination on rows, which give rise to the sequence of column indices. A maximal feasible set A makes a submatrix that with the same rank as matrix M.

Example A.4 (Knapsack)**.** Let the ground set be $E = \{e_1, e_2, \cdots, e_n\}$ and the cost function be $\lambda : E \to \mathbb{R}_{>0}$. Given the budget B for the knapsack problem, the accessible set system is $(E, \mathcal{F})$, where $\mathcal{F} = \{S \subseteq E : \sum_{e \in S} \lambda(e) \leq B\}$. $(E, \mathcal{F})$ corresponding to the knapsack problem is neither a greedoid nor a matroid, but nevertheless still an accessible set system. A maximal feasible set S is the one that cannot include any more element in $E \setminus S$ without violating the budget constraint, and maximal feasible sets may have different lengths.

B Some Simple Propositions

We first formally define α-*approximation regret*, the regret comparing to the α fraction of the optimal solution (as defined in [11]), in the following.

Definition B.1 (α-approximation regret)**.** Let $S^* = \arg\max_{S \in \mathcal{F}} \mathbb{E}\left[f_1(S)\right]$, and denote $\sigma^t = \langle S_0^t, S_1^t, \ldots, S_{m^t}^t \rangle$, where $S^t := S_{m^t}^t$, as the decision sequence selected for any time $t = 1, 2, \cdots, T$. We define the α-approximation regret ($0 < \alpha \leq 1$) as

$$R^\alpha(T) := \alpha T \cdot \mathbb{E}\left[f_1(S^*)\right] - \sum_{t=1}^{T} \mathbb{E}\left[f_t(S^t)\right]. \tag{9}$$

Then, the next two simple propositions show that the α-approximation regret can be derived from the greedy regret and ϵ-quasi greedy regret if the offline greedy algorithm achieves α-approximation solution.

Proposition B.1. *For the greedy sequence σ^{G}, if the maximal feasible set in σ^{G} is an α-approximation for the offline problem, where $\alpha \in (0, 1]$, then $R^\alpha(T) \leq R^{\mathsf{G}}(T)$.*

Proof. For the greedy sequence $\sigma^{\mathsf{G}} := \langle G_0, G_1, \ldots, G_{m^{\mathsf{G}}} \rangle$. Since $G_{m^{\mathsf{G}}}$ is an α-approximation solution, thus $\mathbb{E}\left[f_1(S^{\mathsf{G}})\right] = \overline{f}(S^{\mathsf{G}}) \geq \alpha \overline{f}(S^*) = \mathbb{E}\left[f_1(S^*)\right]$. Therefore, it follows that

$$R^{\mathsf{G}}(T) = T \cdot \mathbb{E}\left[f_1(S^{\mathsf{G}})\right] - \sum_{t=1}^{T} \mathbb{E}\left[f_t(S^t)\right] \geq T \cdot \alpha\, \mathbb{E}\left[f_1(S^*)\right] - \sum_{t=1}^{T} \mathbb{E}\left[f_t(S^t)\right] = R^\alpha(T).$$

□

A similar proposition holds for ϵ-quasi greedy sequences. Among all ϵ-quasi greedy sequences, we use $\sigma^{\mathsf{Q}} := \langle Q_0, Q_1, \ldots, Q_{m^{\mathsf{Q}}} \rangle$ to denote the one with the minimum reward, where m^{Q} is its size.

Proposition B.2. *For an ϵ-quasi greedy sequence $\sigma^{\mathsf{Q}} := \langle Q_0, Q_1, \ldots, Q_{m^{\mathsf{Q}}} \rangle$, if $Q_{m^{\mathsf{Q}}}$ is an α-approximation for the offline problem, where $\alpha \in (0, 1]$, then $R^\alpha(T) \leq R^{\mathsf{Q}}(T)$.*

Proof. Since $Q_{m^{\mathsf{Q}}}$ is an α-approximation, $\mathbb{E}\left[f_1(Q_{m^{\mathsf{Q}}})\right] = \overline{f}(Q_{m^{\mathsf{Q}}}) \geq \alpha \overline{f}(S^*) = \mathbb{E}\left[f_1(S^*)\right]$. Hence, we can get that

$$R^{\mathsf{Q}}(T) = T \cdot \mathbb{E}\left[f_1(Q_{m^{\mathsf{Q}}})\right] - \sum_{t=1}^{T} \mathbb{E}\left[f_t(S^t)\right] \geq T \cdot \alpha\, \mathbb{E}\left[f_1(S^*)\right] - \sum_{t=1}^{T} \mathbb{E}\left[f_t(S^t)\right] = R^\alpha(T).$$

□

C Analysis for OG-UCB in Section 3

In this section, we analyze the greedy regret bound for Algorithm OG-UCB (Algorithm 1 + Subroutine 2). In Appendix C.1, we first derive a general form of the greedy regret bound. Then, in Appendix C.2, we analyze the expected number of playing each arm in the decision frontier, and then obtain the desired greedy regret bound for Algorithm OG-UCB.

C.1 Set Decomposition

We first define some useful notations. For any time t, suppose Algorithm OG-UCB plays a maximal decision sequence $\sigma^t := \langle S_0^t, S_1^t, \ldots, S_{m^t}^t \rangle$ where $S^t := S_{m^t}^t$, and $S_i^t = S_{i-1}^t \cup \{s_i^t\}$ for each i. Denote the greedy sequence as $\sigma^{\mathsf{G}} := \langle G_0, G_1, \ldots, G_{m^{\mathsf{G}}} \rangle$, where $G_i = G_{i-1} \cup \{g_i\}$ for each i. For any time t and any $i = 1, 2, \cdots$, define random event $\mathcal{O}_i^t := \{G_i = S_i^t\}$ (we define $\mathcal{O}_0^t = \{\emptyset = \emptyset\}$ always holds.) which means that OG-UCB finds the same prefix S_i^t as the greedy sequence does. For ease of notation, let $\mathcal{O}_{0:k}^t := \bigcap_{i=0}^k \mathcal{O}_i^t$ for each $k \geq 0$.

Then, we can get the following lemma by De Morgan's laws:

Lemma C.1 (Set-decomposition)**.** *Fix any $k \geq 1$, then $\overline{\bigcap_{i=1}^k \mathcal{O}_i^t} = \left(\overline{\mathcal{O}_1^t}\right) \cup \left(\mathcal{O}_1^t \cap \overline{\mathcal{O}_2^t}\right) \cup \cdots \cup \left(\left(\bigcap_{i=1}^{k-1} \mathcal{O}_i^t\right) \cap \overline{\mathcal{O}_k^t}\right) = \bigcup_{j=1}^k \left(\mathcal{O}_{0:j-1}^t \cap \overline{\mathcal{O}_j^t}\right)$, and each term of its right-hand side is mutually exclusive.*

Notice that $\mathcal{O}_{1:j-1}^t \cap \overline{\mathcal{O}_j^t}$ means that the first $j-1$ prefixes coincide with the greedy sequence. i.e., $G_i = S_i^t$, for $i = 1, \cdots, j-1$; however, at j-th phase, it picks $s_j^t \neq g_j$, that is $s_j^t \in \mathcal{N}_-(G_{j-1})$. Now, we can write the greedy regret as follows.

Lemma C.2. *For any time T, the greedy regret of Algorithm OG-UCB satisfies that:*

$$R^{\mathsf{G}}(T) \leq \sum_{k=1}^{m^{\mathsf{G}}} \mathbb{E}\left[\sum_{t=1}^T \Delta(S^t) \cdot \mathbb{I}\left\{\mathcal{O}_{1:k-1}^t \cap \overline{\mathcal{O}_k^t}\right\}\right].$$

Proof. By definition, $\overline{f}(S) = \mathbb{E}_{\omega_1}[f_1(S)]$ and $\Delta(S) = \overline{f}(G_{m^{\mathsf{G}}}) - \overline{f}(S)$ for any $S \in \mathcal{F}$. Therefore, we have that

$$\begin{aligned} R^{\mathsf{G}}(T) =& T \cdot \mathbb{E}\left[f_1(G_{m^{\mathsf{G}}})\right] - \sum_{t=1}^T \mathbb{E}\left[f_t(S^t)\right] \\ =& T \cdot \overline{f}(G_{m^{\mathsf{G}}}) - \sum_{t=1}^T \mathbb{E}\left[\overline{f}(S^t)\right] \\ =& \mathbb{E}\left[\sum_{t=1}^T \Delta(S^t) \cdot \mathbb{I}\{G_{m^{\mathsf{G}}} \neq S^t\}\right]. \end{aligned}$$

From the definition, we know that if $\bigcap_{i=1}^{m^{\mathsf{G}}} \mathcal{O}_i^t$ occurs, then event $\{G_{m^{\mathsf{G}}} = S^t\}$ is true. As its contrapositive, $\{G_{m^{\mathsf{G}}} \neq S^t\}$ implies that $\overline{\cap_{i=1}^{m^{\mathsf{G}}} \mathcal{O}_i^t}$ occurs. Therefore, we can get

$$\begin{aligned} \mathbb{E}\left[\sum_{t=1}^T \Delta(S^t) \cdot \mathbb{I}\{G_{m^{\mathsf{G}}} \neq S^t\}\right] &\leq \mathbb{E}\left[\sum_{t=1}^T \Delta(S^t) \cdot \mathbb{I}\left\{\overline{\cap_{i=1}^{m^{\mathsf{G}}} \mathcal{O}_i^t}\right\}\right] \\ &\leq \mathbb{E}\left[\sum_{t=1}^T \Delta(S^t) \cdot \mathbb{I}\left\{\bigcup_{k=1}^{m^{\mathsf{G}}} \left(\mathcal{O}_{0:k-1}^t \cap \overline{\mathcal{O}_k^t}\right)\right\}\right] \quad (10) \\ &\leq \mathbb{E}\left[\sum_{t=1}^T \Delta(S^t) \cdot \sum_{k=1}^{m^{\mathsf{G}}} \mathbb{I}\left\{\mathcal{O}_{0:k-1}^t \cap \overline{\mathcal{O}_k^t}\right\}\right] \quad (11) \\ &= \sum_{k=1}^{m^{\mathsf{G}}} \mathbb{E}\left[\sum_{t=1}^T \Delta(S^t) \cdot \mathbb{I}\left\{\mathcal{O}_{0:k-1}^t \cap \overline{\mathcal{O}_k^t}\right\}\right], \end{aligned}$$

where (10) is due to Lemma C.1, and (11) is by the union bound. □

C.2 Proof of Theorem 3.1

We use the following well-known probability inequality.

Fact C.3 (Chernoff-Hoeffding Bound)**.** *$Z_1, Z_2, \ldots, Z_m$ are i.i.d. random variables supported on $[0,1]$ with mean μ. Define the mean estimator as $\hat{Z} = \frac{1}{m}\sum_{i=1}^m Z_m$. For any $a > 0$, it holds that*

$$\mathbb{P}\left[\hat{Z} > \mu + \epsilon\right] \leq \exp\left\{-2\epsilon^2 m\right\}, \quad \mathbb{P}\left[\hat{Z} < \mu - \epsilon\right] \leq \exp\left\{-2\epsilon^2 m\right\}.$$

Recall that the unit gap for any arm $a = e|S$ (defined in (4)) is

$$\Delta(a) = \Delta(e|S) = \begin{cases} \overline{f}(g_S^*|S) - \overline{f}(e|S), & e \neq g_S^* \\ \overline{f}(e|S) - \max_{e' \in \mathcal{N}_-(S)} \overline{f}(e'|S), & e = g_S^* \end{cases}.$$

Define the threshold function for any arm $a = e|S \in \mathcal{A}$ as

$$\theta_t(a) = \frac{6 \ln t}{\Delta(a)^2}, \tag{12}$$

and the random event

$$\mathcal{E}^t(e|S) = \{N_t(e|S) \geq \theta_t(e|S)\} \tag{13}$$

meaning that the arm $e|S$ is *sufficient sampled* at time t.

Lemma C.4 and Lemma C.5 is a technique adapted from [3], which can bound the error probability of the UCB policy for any sufficiently sampled arm.

Lemma C.4. *Consider two arms a_1 and a_2 that generate random variables on the support $[0,1]$, denoted as X_1 and X_2, respectively, and denote $\mu_i = \mathbb{E}[X_i]$ for $i = 1, 2$. Let $\hat{X}_i$ be the average of all i.i.d. samples from X_i (i.e., the empirical mean of X_i), and $N(a_i)$ be the counter of samples. Denote $\hat{X}_{i,N(a_i)}$ be the average of the first $N(a_i)$ samples. W.L.O.G, assume $d = \mu_1 - \mu_2 > 0$. Define function $\mathrm{rad}_t(a) := \sqrt{\frac{3 \ln t}{2N(a)}}$. For $t = 1, 2, \ldots$, if $N(a_1), N(a_2) \geq 1$ and $N(a_2) > \frac{6 \ln t}{d^2}$, the following holds:*

$$\begin{aligned} &\mathbb{P}\left[\hat{X}_{1,N(a_1)} + \mathrm{rad}_t(a_1) \leq \hat{X}_{2,N(a_2)} + \mathrm{rad}_t(a_2)\right] \\ &\leq \mathbb{P}\left[\hat{X}_{1,N(a_1)} \geq \mu_1 - \mathrm{rad}_t(a_1)\right] + \mathbb{P}\left[\hat{X}_{2,N(a_2)} \leq \mu_2 + \mathrm{rad}_t(a_2)\right]. \end{aligned}$$

Proof. Since $N(a_1), N(a_2) \geq 1$ and $N(a_2) > \frac{6 \ln t}{d^2}$, we can get $\mathrm{rad}_t(a_2) < \frac{d}{2}$ and $\mathrm{rad}_t(a_1) > 0$. Thus, $\mu_1 > \mu_2 + 2\,\mathrm{rad}_t(a_2)$. It is easy to verify that event $\hat{X}_{1,N(a_1)} + \mathrm{rad}_t(a_1) < \hat{X}_{2,N(a_2)} + \mathrm{rad}_t(a_2)$ implies at least one of the following events must hold:

$$\hat{X}_{1,N(a_1)} \leq \mu_1 - \mathrm{rad}_t(a_1) \tag{14}$$

$$\hat{X}_{2,N(a_2)} \geq \mu_2 + \mathrm{rad}_t(a_2). \tag{15}$$

Otherwise, assume that both of (14) and (15) are false, then $\hat{X}_{1,N(a_1)} + \mathrm{rad}_t(a_1) > \mu_1 > \mu_2 + 2\,\mathrm{rad}_t(a_2) > \hat{X}_{2,N(a_2)} + \mathrm{rad}_t(a_2)$, which causes a contradiction. Therefore, we can get $\mathbb{P}\left[\hat{X}_{1,N(a_1)} + \mathrm{rad}_t(a_1) < \hat{X}_{2,N(a_2)} + \mathrm{rad}_t(a_2)\right] = \mathbb{P}\left[(14) \text{ or } (15) \text{ is true}\right]$, and the lemma can be concluded by the union bound. $\square$

For any arm $a \in \mathcal{F}$, we denote the upper bound as

$$U_t(a) = \hat{X}(a) + \mathrm{rad}_t(N(a)). \tag{16}$$

As (3) defined in the main text, the counter $N_t(a) = \sum_{i=1}^{t-1} \mathbb{I}_i\{a\}$ is used to denoted the particular $N(a)$ at the beginning of the time step t, where $\mathbb{I}_i\{a\} \in \{0,1\}$ indicates whether arm a is updated at time i. (For OG-UCB, it is always updated once chosen.)

We need the following lemma, which bounds the expected value of the counter.

Lemma C.5. *For any time horizon T and any $k \geq 1$, for any $e \in \mathcal{N}_-(G_{k-1})$, its counter satisfies*

$$\mathbb{E}\left[N_{T+1}(e|G_{k-1})\right] \leq \theta_T(e|G_{k-1}) + \frac{\pi^2}{3} + 1. \tag{17}$$

Proof. Consider a fixed prefix G_{k-1} and an arm $e \in \mathcal{N}_-(G_{k-1})$. Suppose each arm in $\mathcal{N}(G_{k-1})$ is initialized (played once). Line 11 of Subroutine 2 (the UCB policy) indicates that, selecting element $e \in \mathcal{N}_-(G_{k-1})$ for phase k implies that the random event $\{U_t(g_k|G_{k-1}) \leq U_t(e|G_{k-1}\}$ occurs. Then, we can bound the counter of playing $e|G_{k-1}$ by considering two disjoint random events, $\mathcal{E}^t(e|G_{k-1})$ and $\overline{\mathcal{E}^t(e|G_{k-1})}$ (sufficiently sampled and insufficiently sampled, respectively). That is

$$\begin{aligned}
&\mathbb{E}\left[N_{T+1}(e|G_{k-1})\right] \\
\leq &\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\mathcal{O}^t_{0:k-1} \wedge U_t(g_k|G_{k-1}) \leq U_t(e|G_{k-1})\right\}\right] \\
= &\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\mathcal{O}^t_{0:k-1} \wedge \overline{\mathcal{E}^t(e|G_{k-1})} \wedge U_t(g_k|G_{k-1}) \leq U_t(e|G_{k-1})\right\}\right] \\
&+ \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\mathcal{O}^t_{0:k-1} \wedge \mathcal{E}^t(e|G_{k-1}) \wedge U_t(g_k|G_{k-1}) \leq U_t(e|G_{k-1})\right\}\right] \\
\leq &\lceil \theta_T(e|G_{k-1}) \rceil + \sum_{t=1}^{T} \mathbb{P}\left[\mathcal{O}^t_{0:k-1} \wedge \mathcal{E}^t(e|G_{k-1}) \wedge U_t(g_k|G_{k-1}) \leq U_t(e|G_{k-1})\right], \qquad (18)
\end{aligned}$$

where (18) follows from the definition of $\mathcal{E}^t(e|G_{k-1})$. Now we use Lemma C.4, in which arms are $a_1 = g_k|G_{k-1}$ and $a_2 = e|G_{k-1}$; empirical means $\hat{X}_1$, $\hat{X}_2$ are associated with $\hat{X}(g_k|G_{k-1})$ and $\hat{X}(e|G_{k-1})$, respectively; μ_1, μ_2 are used to denote their means; $N(a_1)$ and $N(a_2)$ are their counters. Then, for any t, we have that

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{i=1}^{k-1} \mathcal{O}^t_i \wedge \mathcal{E}^t(e|G_{k-1}) \wedge U_t(g_k|G_{k-1}) \leq U_t(e|G_{k-1})\right] \\
\leq &\mathbb{P}\left[\mathcal{E}^t(e|G_{k-1}) \wedge U_t(g_k|G_{k-1}) \leq U_t(e|G_{k-1})\right] \\
\leq &\sum_{N(a_1)=1}^{t} \mathbb{P}\left[\hat{X}_{1,N(a_1)} \geq \mu_1 - \mathrm{rad}_t(a_1)\right] + \sum_{N(a_2)=\lceil \theta_t(a_2) \rceil}^{t} \mathbb{P}\left[\hat{X}_{2,N(a_2)} \leq \mu_2 + \mathrm{rad}_t(a_2)\right] \\
\leq &\sum_{N(a_1)=1}^{t} t^{-3} + \sum_{N(a_2)=1}^{t} t^{-3} = 2t^{-2}, \qquad (19)
\end{aligned}$$

where (19) holds because of Fact C.3. Summing over $t = 1, \ldots, T$ and using the convergence of Riemann zeta function (i.e., $\sum_{t=1}^{\infty} \frac{1}{t^2} = \frac{\pi^2}{6}$), the proof is completed. $\square$

Theorem C.6 (Restatement of Theorem 3.1)**.** *For any time T, Algorithm* OG-UCB *(Algorithm 1 + Subroutine 2) can achieve the greedy regret*

$$R^{\mathsf{G}}(T) \leq \sum_{a \in \Gamma_-(\sigma^{\mathsf{G}})} \left(\frac{6\Delta^*(a) \cdot \ln T}{\Delta(a)^2} + \left(\frac{\pi^2}{3} + 1\right)\Delta^*(a)\right),$$

where σ^{G} is the greedy decision sequence.

Proof. Now fix any k, and consider each term of the right-hand side in Lemma C.2, that is $\mathbb{E}\left[\sum_{t=1}^{T} \Delta(S^t) \cdot \mathbb{I}\left\{\mathcal{O}^t_{0:k-1} \cap \overline{\mathcal{O}^t_k}\right\}\right]$.

Suppose $\mathcal{O}^t_{0:k-1} \cap \overline{\mathcal{O}^t_k}$ happens, that is the algorithm selects G_{k-1} at the first $k-1$ phases, and then picks $e \in \mathcal{N}_-(G_{k-1})$. By definition of the sunk-cost gap in (5), we can see that the regret at time t is no more than $\Delta^*(e|G_{k-1})$. We we can get:

$$\mathbb{E}\left[\sum_{t=1}^{T} \Delta(S^t) \cdot \mathbb{I}\left\{\mathcal{O}^t_{0:k-1} \cap \overline{\mathcal{O}^t_k}\right\}\right]$$

$$\leq \mathbb{E}\left[\sum_{t=1}^{T} \Delta(S^t) \cdot \mathbb{I}\left\{\mathcal{O}^t_{0:k-1}\right\} \cdot \left(\sum_{e\in\mathcal{N}_-(G_{k-1})} \mathbb{I}_t\left\{e|G_{k-1}\right\}\right)\right] \tag{20}$$

$$\leq \sum_{e\in\mathcal{N}_-(G_{k-1})} \mathbb{E}\left[\sum_{t=1}^{T} \Delta(S^t) \cdot \mathbb{I}\left\{\mathcal{O}^t_{0:k-1}\right\} \cdot \mathbb{I}_t\left\{e|G_{k-1}\right\}\right]$$

$$\leq \sum_{e\in\mathcal{N}_-(G_{k-1})} \Delta^*(e|G_{k-1}) \cdot \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\mathcal{O}^t_{0:k-1}\right\} \cdot \mathbb{I}_t\left\{e|G_{k-1}\right\}\right]$$

$$\leq \sum_{e\in\mathcal{N}_-(G_{k-1})} \Delta^*(e|G_{k-1}) \cdot \mathbb{E}\left[N_{T+1}(e|G_{k-1})\right] \tag{21}$$

where (20) is by the union bound, and (21) is by the definition of $N_{T+1}(e|G_{k-1})$ and $\mathbb{I}\left\{\mathcal{O}^t_{0:k-1}\right\} \leq 1$. Thus, the greedy regret satisfies that

$$R^{\mathsf{G}}(T) \leq \sum_{k=1}^{m^{\mathsf{G}}} \mathbb{E}\left[\sum_{t=1}^{T} \Delta(S^t) \cdot \mathbb{I}\left\{\mathcal{O}^t_{1:k-1} \cap \overline{\mathcal{O}^t_k}\right\}\right] \tag{22}$$

$$\leq \sum_{k=1}^{m^{\mathsf{G}}} \sum_{e\in\mathcal{N}_-(G_{k-1})} \Delta^*(e|G_k) \cdot \mathbb{E}\left[N_T(e|G_{k-1})\right] \tag{23}$$

$$= \sum_{a\in\Gamma_-(\sigma^{\mathsf{G}})} \Delta^*(a) \cdot \mathbb{E}\left[N_T(a)\right] \tag{24}$$

$$\leq \sum_{a\in\Gamma_-(\sigma^{\mathsf{G}})} \left(\frac{6\Delta^*(a) \cdot \ln T}{\Delta(a)^2} + \left(\frac{\pi^2}{3} + 1\right) \Delta^*(a)\right), \tag{25}$$

where (22) is derived from Lemma C.2; (23) is from (21); (24) is due to Definition 3.2; and (25) follows Lemma C.5. Therefore, the theorem is concluded. □

D Analysis for OG-LUCB$_{\epsilon,\delta}$ and OG-LUCB-R in Section 4

In this section, we provide the analysis of Algorithm OG-LUCB$_{\epsilon,\delta}$ (Algorithm 1 + Subroutine 3) and Algorithm OG-LUCB-R (Algorithm 4). In the following, we first derive the probability inequality of confidence events. Second, we analyze the exploration time in Appendix D.1. Assuming that the total time horizon T is known, we provide an $O(\log T)$ ϵ-quasi greedy regret in Appendix D.2. Finally, we show that we can remove the assumption of knowing T and still obtain an $O(\log T)$ regret bound (by Algorithm OG-LUCB-R).

Given any arm $a = e|S \in \mathcal{A}$, define event $\mathcal{C}^t(e|S) = \left\{\left|\hat{X}_t(e|S) - X(e|S)\right| < \mathrm{rad}_t(e|S)\right\}$ (confidence bound holds). For any prefix S, define $\mathcal{C}^t(S) = \bigcup_{e\in\mathcal{N}(S)} \mathcal{C}^t(e|S)$ for all the arms $e|S$.

Recall that for any S, $e \in \mathcal{N}(S)$, by definition

$$\Delta(e|S_k) = \begin{cases} \overline{f}(g^*_S|S) - \overline{f}(e|S), & e \neq g^*_S \\ \overline{f}(e|S) - \max_{a\in\mathcal{N}(S)\setminus\{g^*_S\}} \overline{f}(a|S), & e = g^*_S \end{cases},$$

and the hardness coefficient with the prefix S as

$$H^{\epsilon}_S := \sum_{e\in\mathcal{N}(S)} \frac{1}{\max\left\{\Delta(e|S)^2, \epsilon^2\right\}}.$$

Lemma D.1. *Fix any $\delta \in (0,1)$. Suppose $\mathrm{rad}_t(s) = \sqrt{\frac{\ln(4Wt^3/\delta)}{2N_t(s)}}$. For any prefix S, $\bigcup_{t=1}^{\infty} \mathcal{C}_t(S)$ holds with probability at least $1-\delta$.*

Proof. From Fact C.3, we know that for any $e \in \mathcal{N}(S)$,

$$\mathbb{P}\left[\mathcal{C}^t(e|S)\right] = \mathbb{P}\left[\left|\hat{X}_t(e|S) - X(e|S)\right| < \mathrm{rad}_t(e|S), N_t(e|S) = 1, \cdots, t\right]$$

$$\geq 1 - 2 \cdot t \cdot \frac{\delta}{4Wt^3} = 1 - \frac{\delta}{2Wt^2}.$$

Then, by summing all time steps t and all elements $e \in \mathcal{N}(S)$ ($|\mathcal{N}(S)| \leq W$), we can conclude that

$$\begin{aligned}\mathbb{P}\left[\bigcup_{t=1}^{\infty} \mathcal{C}^t(S)\right] \geq 1 - \sum_{t=1}^{\infty} \sum_{e \in \mathcal{N}(S)} \frac{\delta}{2Wt^2} \geq & 1 - \sum_{t=1}^{\infty} \frac{\delta}{2t^2} \\ \geq & 1 - \frac{\pi^2 \delta}{12} \qquad (26) \\ \geq & 1 - \delta,\end{aligned}$$

where (26) is by Riemann zeta function. □

D.1 Exploration time for OG with a first-explore-then-exploit policy (OG-LUCB$_{\epsilon,\delta}$)

Since we use a specialized version of CLUCB-PAC in [9] as $\mathrm{MaxOracle}$ to explore the top-one element in the support of $[0, 1]$ for each phase, and henceforth our analysis starts from the result of sample complexity of CLUCB-PAC by setting $R = \frac{1}{2}$ and $\mathrm{width}(\mathcal{M}) = 2$.

In Algorithm OG-LUCB$_{\epsilon,\delta}$, for any arm $a = e|S \in \mathcal{A}$, denote $N_t^{\mathsf{E}}(a)$ as the counter of playing arm a during the exploration stage at the beginning of time step t. The algorithm turns from the exploration to the exploitation some time and the counter will not change, therefore we may use $N_{\infty}^{\mathsf{E}}(a)$ to obtain its final value. For any prefix S, denote arms $A = \{e|S : \forall e \in \mathcal{N}(S)\}$. Denote $t_0(S) := \sum_{e|S \in A} N_{t_0}^{\mathsf{E}}(e|S)$ as the total exploration time of all elements in $\mathcal{N}(S)$, such that for any time $t \geq t_0(S)$, $I^{\mathsf{exploit}}(A)$ is initialized.

Notice that we use CLUCB-PAC for each phase, therefore the following lemma can be adapted from the intermediate step in proving Theorem 5 of [9].

Lemma D.2. *For any phase $k \geq 0$, fix prefix S_k. Suppose $\epsilon \geq 0$, $\delta \in (0, 1)$, and $\bigcup_{t=1}^{\infty} \mathcal{C}^t(S_k)$ holds. If it goes to the exploitation stage (Line 12 of Subroutine 3) with $I^{\mathsf{exploit}}(A) = s^*$ at time $t_0 = t_0(S_k)$, then:*

1. *$f(s^*|S_k) \geq f(g^*_{S_k}|S_k) - \epsilon$;*
2. *In addition, for any $s \in \mathcal{N}(S_k)$,*

$$N_{t_0}^{\mathsf{E}}(s|S_k) \leq \min\left\{\frac{18}{\Delta(s|S_k)^2}, \frac{16}{\epsilon^2}\right\} \ln\left(4Wt_0^3/\delta\right) + 1. \qquad (27)$$

Notice that event $\bigcup_{t=1}^{\infty} \mathcal{C}^t(S_k)$ holds with probability at least $1 - \delta$, which is guaranteed by Lemma D.1. From the above lemma, we may further derive the following bound for $t_0(S_k)$.

Lemma D.3. *With the same setting as Lemma D.2. For $t_0 = t_0(S_k)$, denote $t_c = t_c(S_k) := 499H^{\epsilon}_{S_k} \ln(4WH^{\epsilon}_{S_k}/\delta) + 2W$ then we can get:*

1. *$t_0 \leq t_c$.*
2. *$\ln\left(4Wt_c^3/\delta\right) \leq 7 \ln\left(1996WH^{\epsilon}_{S_k}/\delta\right)$.*

Proof. **Property (1):** First of all, (1) holds trivially if $W \geq \frac{t_0}{2}$, thus we only need to show the case $W < \frac{t_0}{2}$.

Since $t_0 = \sum_{e \in \mathcal{N}(S_k)} N_{t_0}^{\mathsf{E}}(e|S_k)$, it can be implied from Lemma D.2 that, with probability at least $1 - \delta$,

$$t_0 \leq 18H^{\epsilon}_{S_k} \ln(4Wt_0^3/\delta) + W. \qquad (28)$$

We assume $t_0 = CH^{\epsilon}_{S_k} \ln(4WH^{\epsilon}_{S_k}/\delta) + W$, for some constant $C > 0$. When $W < \frac{t_0}{2}$, $t_0 < 2CH^{\epsilon}_{S_k} \ln(4WH^{\epsilon}_{S_k}/\delta)$. Then, rewrite (28) as

$$t_0 \leq W + 18H^{\epsilon}_{S_k} \ln(4W/\delta) + 54H^{\epsilon}_{S_k} \ln(t_0)$$

$$
\begin{aligned}
&<W + 18H^{\epsilon}_{S_k}\ln(4W/\delta) + 54H^{\epsilon}_{S_k}\ln(2CH^{\epsilon}_{S_k}\ln(4WH^{\epsilon}_{S_k}/\delta)) \\
&\leq W + 18H^{\epsilon}_{S_k}\ln(4W/\delta) + 54H^{\epsilon}_{S_k}\left(\ln(2C) + \ln(H^{\epsilon}_{S_k})\right) + 54H^{\epsilon}_{S_k}\ln(4WH^{\epsilon}_{S_k}/\delta) \\
&\leq W + 72H^{\epsilon}_{S_k}\ln(4WH^{\epsilon}_{S_k}/\delta) + 54H^{\epsilon}_{S_k}\left(\ln(2C)\ln(4WH^{\epsilon}_{S_k}/\delta) + \ln(4WH^{\epsilon}_{S_k}/\delta)\right) \\
&\leq W + (126 + 54\ln 2C)\, H^{\epsilon}_{S_k}\ln(4WH^{\epsilon}_{S_k}/\delta). \qquad (29)
\end{aligned}
$$

Solve $126 + 54\ln 2C < C$, and we can get the minimum integer solution $C = 499$. When $C \geq 499$, from (29), $t_0 < W + CH^{\epsilon}_{S_k}\ln(4WH^{\epsilon}_{S_k}/\delta) = t_0$, which cause a contradiction. Thus, we can conclude that $t_0 \leq 499H^{\epsilon}_{S_k}\ln(4WH^{\epsilon}_{S_k}/\delta) + 2W$.

Property (2): We can simplify $\ln\left(4Wt_c^3/\delta\right)$ in (27) as follows.

From Property 1, since $\ln(a+b) \leq \ln(a) + \ln(b)$ for all $a, b \geq e$, we have

$$
\begin{aligned}
\ln(t_c) &\leq \ln\left(499H^{\epsilon}_{S_k}\ln(4WH^{\epsilon}_{S_k}/\delta)\right) + \ln(4W) \qquad (30)\\
&\leq \ln(499H^{\epsilon}_{S_k}) + \ln\ln(4WH^{\epsilon}_{S_k}/\delta) + \ln(4W) \\
&\leq \ln(499H^{\epsilon}_{S_k}) + 2\ln(4W) + \ln\left(H^{\epsilon}_{S_k}\right) + \ln(1/\delta),
\end{aligned}
$$

then

$$
\begin{aligned}
\ln\left(4Wt_c^3/\delta\right) &\leq \ln(4W) + \ln(1/\delta) + 3\ln(t_c) \\
&\leq 7\ln(4W) + 4\ln(1/\delta) + 6\ln\left(499H^{\epsilon}_{S_k}\right) \\
&\leq 7\ln\left(1996WH^{\epsilon}_{S_k}/\delta\right),
\end{aligned}
$$

which ends the proof. $\square$

Define random event iteratively, for any $k \geq 1$, given prefix S_{k-1},

$$
\mathcal{Q}_k(S_{k-1}) = \left\{\overline{f}(s_k|S_{k-1}) \geq \overline{f}(g^*_{S_{k-1}}|S_{k-1}) - \epsilon\right\}, \qquad (31)
$$

where $s_k = I^{\mathsf{exploit}}(A)$ and $A = \{e|S_{k-1} : \forall e \in \mathcal{N}(S_{k-1})\}$.

Theorem D.4 (Restatement of Theorem 4.1)**.** *Given any $\epsilon \geq 0$ and $\delta \in (0,1)$, suppose after the total exploration time $T^{\mathsf{E}} = T^{\mathsf{E}}(\delta)$, Algorithm OG-LUCB$_{\epsilon,\delta}$ sticks to a stable sequence $\sigma = \langle S_0, S_1, \cdots, S_{m'}\rangle$. With probability at least $1 - m\delta$ (m is the largest length of a feasible set), the following three claims hold:*

1. *σ is an ϵ-quasi greedy sequence;*
2. *Let $t_c(S) := 499H^{\epsilon}_S\ln(4WH^{\epsilon}_S/\delta) + 2W$. For any arm $e|S$ in the decision frontier $\Gamma(\sigma)$,*

$$
N^{\mathsf{E}}_{\infty}(e|S) \leq \min\left\{\frac{18}{\Delta(e|S)^2}, \frac{16}{\epsilon^2}\right\}\ln\left(4Wt_c^3(S)/\delta\right) + 1. \qquad (32)
$$

3. *The total exploration time T^{E} satisfies that*

$$
T^{\mathsf{E}} \leq \sum_{e|S\in\Gamma(\sigma)} N^{\mathsf{E}}_{\infty}(e|S) \leq 127\sum_{k=0}^{m'-1} H^{\epsilon}_S\ln\left(1996WH^{\epsilon}_S/\delta\right). \qquad (33)
$$

Proof. **Property (1):** Similar to the set decomposition in Lemma C.1, by De Morgan's laws, we know that

$$
\overline{\bigcap_{i=1}^{m'}\mathcal{Q}^t_i(S_{i-1})} = \left(\overline{\mathcal{Q}^t_1(S_0)}\right) \cup \left(\mathcal{Q}^t_1(S_0) \cap \overline{\mathcal{Q}^t_2(S_1)}\right) \cup \cdots \cup \left(\left(\bigcap_{i=1}^{m'-1}\mathcal{Q}^t_i(S_{i-1})\right) \cap \overline{\mathcal{Q}^t_k(S_{m'-1})}\right).
$$

By Lemma D.2, we know that for $k = 1, 2, \cdots, m'$, the k-th term above satisfies

$$
\mathbb{P}\left[\left(\bigcap_{i=1}^{k-1}\mathcal{Q}^t_i(S_{i-1})\right) \cap \overline{\mathcal{Q}^t_k(S_{k-1})}\right] \leq \mathbb{P}\left[\overline{\mathcal{Q}^t_k(S_{k-1})}\right] \leq \delta. \qquad (34)
$$

Therefore, by union bound and $m' \le m$, we have that

$$\mathbb{P}\left[\bigcap_{i=1}^{m'} \mathcal{Q}_i^t(S_{k-1})\right] = 1 - \mathbb{P}\left[\overline{\bigcap_{i=1}^{m'} \mathcal{Q}_i^t(S_{k-1})}\right] \le 1 - \sum_{k=1}^{m'} \delta \le 1 - m\delta,$$

and we prove that σ is an ϵ-quasi greedy sequence.

Property (2): Assume that $\bigcap_{i=1}^{m'} \mathcal{Q}_i^t(S_{i-1})$ holds. Since we denote $t_c(S) = 499 H_S^\epsilon \ln(4WH_S^\epsilon/\delta) + 2W$, applying Property 1 of Lemma D.3, we can get $t_0(S_{k-1}) \le t_c(S_{k-1})$. Then, use Property 2 of Lemma D.2, and it follows immediately that

$$N_\infty^{\mathsf{E}}(e|S_{k-1}) \le \min\left\{\frac{18}{\Delta(e|S_{k-1})^2}, \frac{16}{\epsilon^2}\right\} \ln\left(4Wt_c^3(S_{k-1})/\delta\right) + 1, \tag{35}$$

where $e|S_{k-1}$ is any arm occurred in the decision frontier $\Gamma(\sigma)$.

Property (3): It is easy to see that the total exploration time $T^{\mathsf{E}} \le \sum_{e|S\in\Gamma(\sigma)} N_\infty^{\mathsf{E}}(e|S)$. Sum up each phase $k = 0, 1, \cdots, m'-1$ and each element $e \in \mathcal{N}(S_k)$, and use $\ln\left(4Wt_c^3(S_{k-1})\right) \le 7\ln\left(1996WH_{S_k}^\epsilon/\delta\right)$ by Property 2 of Lemma D.3. Therefore we can get:

$$\begin{aligned}
T^{\mathsf{E}} \le \sum_{e|S\in\Gamma(\sigma)} N_\infty^{\mathsf{E}}(e|S) &\le \sum_{k=0}^{m'-1} \sum_{e\in\mathcal{N}(S_k)} N_\infty^{\mathsf{E}}(e|S_k) \\
&\le \sum_{k=0}^{m'-1} \left(126 H_{S_k}^\epsilon \ln\left(1996WH_{S_k}^\epsilon/\delta\right) + |\mathcal{N}(S_k)|\right) \\
&\le 127 \sum_{k=0}^{m'-1} H_{S_k}^\epsilon \ln\left(1996WH_{S_k}^\epsilon/\delta\right).
\end{aligned} \tag{36}$$

□

D.2 Time Horizon T is Known

We first assume that we know the total time horizon T. In this case, we may let $\delta = \frac{1}{T}$ in OG-LUCB$_{\epsilon,\delta}$, and we can obtain the following theorem.

Theorem D.5 (Restatement of Theorem 4.2)**.** *Given any $\epsilon \ge 0$. When total time T is known, in Algorithm OG-LUCB$_{\epsilon,\delta}$, we associate $\delta = \frac{1}{T}$. Suppose $\sigma = \langle S_0, S_1, \cdots, S_{m'}\rangle$ is the sequence selected by Algorithm OG-LUCB$_{\epsilon,\delta}$ at time T, and define function*

$$R^{\mathsf{Q},\sigma}(T) := \sum_{e|S\in\Gamma(\sigma)} \Delta^*(e|S) \min\left\{\frac{127}{\Delta(e|S)^2}, \frac{113}{\epsilon^2}\right\} \ln\left(1996WH_S^\epsilon T\right) + \Delta_{\max} m, \tag{37}$$

where m is the largest length of a feasible set and H_S^ϵ is defined in (7). Then, the ϵ-quasi regret satisfies

$$R^{\mathsf{Q}}(T) \le R^{\mathsf{Q},\sigma}(T) = O\left(\frac{Wm\Delta_{\max}}{\max\{\Delta^2, \epsilon^2\}} \log T\right), \tag{38}$$

where Δ is the minimum unit gap.

Proof. First, we claim that

$$R^{\mathsf{Q}}(T) \le \sum_{e|S\in\Gamma(\sigma)} \Delta^*(e|S) \left(\min\left\{\frac{18}{\Delta(e|S)^2}, \frac{16}{\epsilon^2}\right\} \ln\left(4Wt_{c,T}^3(S)\cdot T\right) + 1\right) + \Delta_{\max} m, \tag{39}$$

where $t_{c,T}(S) = 499 H_S^\epsilon \ln(4WH_S^\epsilon \cdot T) + 2W$ for any S.

From Property 1 of Theorem D.4 , we know that with probability $1 - \frac{m}{T}$, the sequence σ is a stable ϵ-greedy sequence, i.e., all confidence events $\bigcap_{i=1}^{m'} \mathcal{Q}_i^t(S_{k-1})$ hold.

We may sum up exploration numbers of the arms ($N_\infty^{\mathsf{E}}(\cdot)$'s) in Property 2 of Theorem D.4, so that the first term of (39) can be easily derived. For the second term of (39), it is because $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ pays at most regret $\Delta_{\max}T$ with probability $\frac{m}{T}$, when any confidence events fails, which contributes at most $\Delta_{\max}m$ to the regret.

Then, apply Property 2 of Lemma D.3 and $\delta = \frac{1}{T}$, i.e., $\ln\left(4Wt_{c,T}^3(S)\cdot T\right) \le 7\ln\left(1996WH_{S_k}^\epsilon/\delta\right)$, and henceforth it is obvious to see that

$$R^{\mathsf{Q}}(T) \le R^{\mathsf{Q},\sigma}(T) = \sum_{e|S\in\Gamma(\sigma)} \Delta^*(e|S)\min\left\{\frac{127}{\Delta(e|S)^2}, \frac{113}{\epsilon^2}\right\}\ln\left(1996WH_S^\epsilon T\right) + \Delta_{\max}m.$$

Furthermore, if we replace $\Delta^*(e|S)$ with the maximum gap $\Delta_{\max}$, and $\Delta(e|S)$ with the minimum unit gap Δ, it is easily derived that $R^{\mathsf{Q}}(T) \le R^{\mathsf{Q},\sigma}(T) = O(\frac{Wm\Delta_{\max}}{\max\{\Delta^2,\epsilon^2\}}\log T)$, where W is the largest width and m is the largest length. ☐

Notice that, from the above theorem, the ϵ-quasi regret is bounded within $O(\log T)$ for any time horizon T, even though T may be less than the total exploration time $T^{\mathsf{E}} = T^{\mathsf{E}}\left(\frac{1}{T}\right)$. It is because $N_\infty^{\mathsf{E}}(\cdot)$ is the upper bound approaching the infinite time, and the regret is a non-decreasing function with the time horizon T. Therefore, the above regret is satisfied for $T < T^{\mathsf{E}}\left(\frac{1}{T}\right)$ as well.

D.3 Time Horizon T is not Known

When the time horizon T is not known, we use Algorithm OG-LUCB-R (Algorithm 4) which restarts the internal $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ for different epochs. The following lemma is useful in proving Theorem D.7.

Lemma D.6. *For any $i = 1, 2, \cdots, k$, for any $c_i, b_i > 0$ and $\phi_i = e^{2^i}$,*

$$\sum_{i=1}^{k}\left(c_i\cdot\ln(\phi_i) + b_i\right) \le 4\cdot\left(\max_{i=1,\cdots,k} c_i\right)\cdot\ln(\phi_{k-1}) + k\cdot\left(\max_{i=1,\cdots,k} b_i\right).$$

Proof. Since $\phi_i = e^{2^i}$, then $\ln(\phi_i) = 2^i$, an we can get

$$\sum_{i=1}^{k}\ln\left(\phi_i\right) = 2^1 + 2^2 + \cdots + 2^k < 4\times 2^{k-1} = 4\ln\left(\phi_{k-1}\right).$$

Therefore, it follows immediately that

$$\begin{aligned}\sum_{i=1}^{k}\left(c_i\cdot\ln(\phi_i) + b_i\right) \le& \left(\max_{i=1,\cdots,k} c_i\right)\sum_{i=1}^{k}\ln(\phi_i) + k\cdot\left(\max_{i=1,\cdots,k} b_i\right)\\ \le& 4\cdot\left(\max_{i=1,\cdots,k} c_i\right)\cdot\ln(\phi_{k-1}) + k\cdot\left(\max_{i=1,\cdots,k} b_i\right).\end{aligned}$$

☐

Theorem D.7 (Restatement of Theorem 4.3)**.** *Given any $\epsilon \ge 0$. Use ϕ_ℓ and τ_ℓ defined in (8), and function $R^{\mathsf{Q},\sigma}(T)$ defined Theorem 4.2. In Algorithm OG-LUCB-R, suppose $\sigma^{(\ell)} = \langle S_0^{(\ell)}, S_1^{(\ell)}, \cdots, S_{m^{(\ell)}}^{(\ell)}\rangle$ is the sequence selected by the end of ℓ-th epoch of $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$, where $m^{(\ell)}$ is its length. For any time T, denote the final epoch as $K = K(T)$ such that $\tau_{K-1} < T \le \tau_K$, and the ϵ-quasi greedy regret satisfies that*

$$R^{\mathsf{Q}}(T) \le \sum_{\ell=1}^{K} R^{\mathsf{Q},\sigma^{(\ell)}}(\phi_\ell). \tag{40}$$

Furthermore, we can get

$$R^{\mathsf{Q}}(T) \leq 4 \cdot \left(\max_{\ell=1,\cdots,K} c_\ell\right) \cdot \ln(T) + \left(\max_{\ell=1,\cdots,K} b_\ell\right) \cdot \log_2\left(2\ln T\right) \tag{41}$$
$$= O\left(\frac{Wm\Delta_{\max}}{\max\{\Delta^2, \epsilon^2\}} \log T\right), \tag{42}$$

where

$$c_\ell := \sum_{e|S \in \Gamma(\sigma^{(\ell)})} \Delta^*(e|S) \min\left\{\frac{127}{\Delta\left(e|S\right)^2}, \frac{113}{\epsilon^2}\right\},$$
$$b_\ell := \sum_{e|S \in \Gamma(\sigma^{(\ell)})} \Delta^*(e|S) \min\left\{\frac{127}{\Delta\left(e|S\right)^2}, \frac{113}{\epsilon^2}\right\} \ln\left(1996WH_S^\epsilon\right) + \Delta_{\max} m,$$

and Δ is the minimum unit gap.

Proof. Algorithm OG-LUCB-R restarts the internal OG-LUCB$_{\epsilon,\delta}$ for each epoch. Since ϵ-quasi regret for epoch ℓ during time $(\tau_{\ell-1}, \tau_\ell]$ is no more than $R^{\mathsf{Q},\sigma^{(\ell)}}(\phi_\ell)$, then the ϵ-quasi greedy regret in (40) follows naturally by accumulating each interval.

For any ℓ, $\phi_{\ell-1} \leq \tau_{\ell-1}, \tau_\ell = \sum_{i=1}^{\ell} e^{2^i} < 2e^{2^\ell} = 2\phi_\ell$. Since $\tau_{K-1} < T \leq \tau_K$, we have $\phi_{K-1} < T < 2\phi_K$ and $K < \log_2(2\ln T)$.

Furthermore, we can get

$$\sum_{\ell=1}^{K} R^{\mathsf{Q},\sigma^{(\ell)}}(\phi_\ell) \leq \sum_{\ell=1}^{K} \left(c_\ell \ln\left(\phi_i\right) + b_\ell\right) \tag{43}$$
$$\leq 4 \cdot \left(\max_{\ell=1,\cdots,K} c_\ell\right) \cdot \ln(\phi_{K-1}) + \left(\max_{\ell=1,\cdots,K} b_\ell\right) \cdot K \tag{44}$$
$$\leq 4 \cdot \left(\max_{\ell=1,\cdots,K} c_\ell\right) \cdot \ln(T) + \left(\max_{\ell=1,\cdots,K} b_\ell\right) \cdot \log_2\left(2\ln T\right), \tag{45}$$

where (43) is by definition of c_i and b_i, and the form of $R^{\mathsf{Q},\sigma^{(\ell)}}$ defined (37); (44) is from Lemma D.6; and (45) is because $\phi_{K-1} < T$ and $K < \log_2(2\ln T)$. Use the similar technique in the proof of Theorem D.5, and we know that (45) is also in $O\left(\frac{Wm\Delta_{\max}}{\max\{\Delta^2,\epsilon^2\}} \log T\right)$. Therefore, the theorem is concluded. $\square$

E Proof of Lower Bound in Section 5

In this section, we construct an instance of the prize-collecting problem, and provide its theoretical analysis. We first recall the *prize-collecting problem* defined in the main text as follows.

Problem. Consider m bandits, each of which has W elements. For each bandit $i = 1, 2, \ldots, m$, denote set $E_i = \{e_{i,1}, e_{i,2}, \ldots, e_{i,W}\}$. In this problem, the player needs to select one element from each bandit in order. The accessible set system is defined as $(E, \mathcal{F})$, where $E = \bigcup_{i=1}^m E_i$, $\mathcal{F} = \cup_{i=1}^m \mathcal{F}_i \cup \{\emptyset\}$, and $\mathcal{F}_i = \{S \subseteq E : |S| = i, \forall k : 1 \leq k \leq i, |S \cap E_k| = 1\}$. The reward function $f : \mathcal{F} \times \Omega \to [0, m]$ is non-decreasing in the first parameter, where the form of f is unknown to the player. Let the minimum unit gap be

$$\Delta := \min\left\{\overline{f}(g_S^*|S) - \overline{f}(e|S) : \forall S \in \mathcal{F}, \forall e \in \mathcal{N}_-(S)\right\} > 0, \tag{46}$$

where the value of Δ is also unknown to the player. The objective of the player is to minimize the greedy regret.

Instance $\mathcal{P}$. We construct a problem instance $\mathcal{P}$ as follows.

We arbitrarily pick the greedy decision sequence $\sigma^{\mathsf{G}} = \langle G_0, G_1, \ldots, G_m \rangle \in \mathcal{F}^{m+1}$ where $G_0 = \emptyset$ and $G_i = G_{i-1} \cup \{e_i^{\mathsf{G}}\}$ for each i.

Assume that $0 < \mu_1 < \mu_2 < \mu_3 < 1$, and $\Delta := \mu_2 - \mu_1 > 0$. Consider that the environment's randomness comes from Bernoulli random variables $\omega_t = \left(\omega_{t,1}^1, \omega_{t,2}^1, \cdots, \omega_{t,1}^m, \omega_{t,2}^m\right) \in \{0,1\}^{2m} = \Omega$, with $\mathbb{E}[\omega_{t,1}^i] = \mu_1$ (low prize) for any $i = 1, 2, \cdots, m$; $\mathbb{E}[\omega_{t,2}^i] = \mu_2$ (medium prize) for $i = 1, 2, \cdots, m-1$, and $\mathbb{E}[\omega_{t,2}^m] = \mu_3$ (high prize). For convenience, given a feasible set S, we define indicator $\mathbb{I}_i^{\mathsf{G}} := \mathbb{I}\{e_i^{\mathsf{G}} \in S\}$ for each i. The exact form of reward function is $f(S, \omega_t) := \sum_{i=1}^m f^{(i)}(S, \omega_t)$, where for each $i = 1, 2, \cdots, m$,

$$f^{(i)}(S, \omega_t) = \begin{cases} \omega_{t,1}^i \overline{\mathbb{I}_i^{\mathsf{G}}} + \omega_{t,2}^i \mathbb{I}_i^{\mathsf{G}}, & G_{i-1} \subseteq S \\ \omega_{t,1}^i, & \text{otherwise} \end{cases}. \tag{47}$$

It is only accessed as a value oracle $f_t(S) := f(S, \omega_t)$ with a given feasible set S, and *the player does not know the form of the reward function*. For example, at time t, provided that we have already selected the greedy prefix G_{i-1} for the first $i-1$ phases (that is $\cup_{j=1}^{i-1}\{e_i^{\mathsf{G}}\} \subseteq S$), if we choose the greedy element e_i^{G} at phase i (that is $\mathbb{I}_i^{\mathsf{G}} = 1$), then the marginal reward $f^{(i)}(S, \omega_t) = \omega_{t,2}^i$, and the feedback for arm $e_i^{\mathsf{G}}|G_{i-1}$ is $f_t(e_i^{\mathsf{G}}|G_{i-1}) = f^{(i)}(S, \omega_t) = \omega_{t,2}^i$; otherwise, we will get $\omega_{t,1}^i$ for choosing the sub-optimal element. Moreover, only if all elements along the greedy sequence are chosen, that is $S = G$, can we get the marginal reward $f^{(m)}(S, \omega_t) = \omega_{t,2}^m$ for the last phase, where $\mathbb{E}[\omega_{t,2}^m] = \mu_3$.

Collecting prizes means that the player should find the greedy sequence to gain medium prizes ($\mathbb{E}[\omega_{t,2}^i] = \mu_2 > \mu_1 = \mathbb{E}[\omega_{t,1}^i]$) for the first $m-1$ phases and achieve the high prize ($\mathbb{E}[\omega_{t,2}^m] = \mu_3$) for the last phase. Since the maximal decision sequence is m phases, it is easy to infer that, the minimum reward and maximum reward in expectation are $m\mu_1$ and $(m-1)\mu_2 + \mu_3$ respectively, and henceforth the maximum gap is $\Delta_{\max} = m\Delta + (\mu_3 - \mu_2)$. When the player mistakenly chooses a wrong element for some phase, denote the minimum gap as $\Delta_{\min} = \Delta + (\mu_3 - \mu_2)$, and $\Delta_{\min}$ is the minimum penalty incurred.

Denote greedy arms as $\mathcal{A}^{\mathsf{G}} = \{e_i^{\mathsf{G}}|G_{i-1} : \forall i = 1, 2, \cdots, W\}$. We say an algorithm is *consistent*, if the sum of playing all arms $a \in \mathcal{A} \setminus \mathcal{A}^{\mathsf{G}}$ is in $o(T^\eta)$, for any $\eta > 0$, i.e., $\mathbb{E}[\sum_{a \in \mathcal{A} \setminus \mathcal{A}^{\mathsf{G}}} N_T(a)] = o(T^\eta)$.

E.1 Lower Bound $\Omega\left(\frac{mW\Delta_{\max}}{\Delta^2}\log T\right)$

In this subsection, we set $\mu_3 - \mu_2$ to be a constant, and derive the $\Omega\left(\frac{mW\Delta_{\min}}{\Delta^2}\log T\right)$ greedy regret lower bound.

Fix any $\ell = 1, 2, \cdots, m-1$. As is illustrated in Figure 1, there exists some feasible sequence σ that coincides with the greedy sequence σ^{G} for the first ℓ arms, and deviates from it for the next $d := m - \ell$ arms, that is $\sigma = \langle G_0, G_1, \cdots, G_\ell, S_{\ell+1}, \cdots, S_m \rangle$, and $S_{\ell+1} \neq G_\ell \cup \{e_i^{\mathsf{G}}\}$. For convenience, we refer those d arms as $a_1, a_2, \ldots, a_d$, and for any fixed time $T \geq T_0$ (T_0 is sufficiently large), we use $\vec{N} = (N(a_1), N(a_2), \cdots, N(a_d))$ to denote counters of $a_1, \ldots, a_d$, respectively, where each $N(a_k) := N_T(a_k)$. Obviously, $N(a_1) \geq N(a_2) \geq \cdots \geq N(a_d)$. Among all those sequences, since we have W options for each arm a_k after a_{k-1}, by Pigeonhole principle, there exists one sequence satisfying that

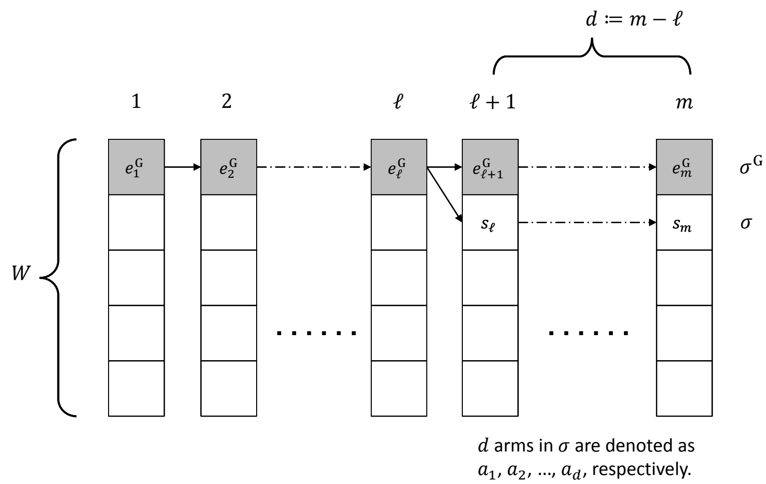
$$N(a_2) \leq \frac{1}{W} N(a_1),\ N(a_3) \leq \frac{1}{W} N(a_2),\ \cdots,\ \text{and } N(a_d) \leq \frac{1}{W} N(a_{d-1}). \tag{48}$$

W.L.O.G., assume that σ satisfies condition (48).

In Lemma E.1, we will show that when d is large enough, the number of playing a_1 is $\Omega\left(\frac{1}{(\mu_2 - \mu_1)^2}\log T\right)$.

Lemma E.1. *Assume that $\frac{1}{4} < \mu_1 < \mu_2 = \frac{1}{2} < \mu_3 = \frac{3}{4}$. Suppose $W \geq 2$, and integer d satisifies that $d \geq \log_W\left(\frac{2W}{3(\mu_2 - \mu_1)^2}\right)$, (i.e., $(\mu_2 - \mu_1)^2 \geq \frac{2}{3W^{d-1}}$). For any consistent algorithm, the number*

Figure 1: Illustration of the sequence σ.



of playing arm a_1 in the sequence σ satisfies that

$$\lim_{t \to +\infty} \frac{\mathbb{E}[N_t(a_1)]}{\ln t} \geq \frac{0.39}{\mathrm{KL}(\mu_1 || \mu_2)},$$

where $\mathrm{KL}(p||q) = p \ln \frac{p}{q} + (1-p) \ln \frac{1-p}{1-q}$ *is Kullback-Leibler divergence of Bernoulli random variables.*

Proof. We keep all arms except $a_1, a_2, \ldots, a_d$ by default, and perturb those arms to construct the following two hypotheses:

$$\text{Null hypothesis } H_\sigma^0: \qquad \overline{f}(a_i) = \mu_1, i \in \{1, \cdots, d-1\}, \quad \overline{f}(a_d) = \mu_1; \tag{49}$$

$$\text{Alternative hypothesis } H_\sigma': \qquad \overline{f}(a_i) = \lambda, i \in \{1, \cdots, d-1\}, \quad \overline{f}(a_d) = \mu_3, \tag{50}$$

where $\lambda \in (\mu_2, 1]$ is some constant to be determined later (in (51)). Note that σ^{G} is its greedy sequence in H_σ^0, while σ is the greedy sequence in H_σ'. For simplicity, we use $\mathbb{P}[\cdot]$ and $\mathbb{P}'[\cdot]$, $\mathbb{E}[\cdot]$ and $\mathbb{E}'[\cdot]$ to denote the probability and expectation under H_σ^0 and H_σ', respectively.

Fix any $\gamma \in (\frac{2}{3}, 1)$, and let λ be some constant such that

$$\mu_2 < \lambda \leq 1, \text{ and } |\mathrm{KL}(\mu_1||\lambda) - \mathrm{KL}(\mu_1||\mu_2)| \leq \gamma \, \mathrm{KL}(\mu_1||\mu_2). \tag{51}$$

Define event $\mathcal{N} := \left\{ N_{a_1} < \frac{1-\gamma}{\mathrm{KL}(\mu_1||\lambda)} \ln T \right\}$.

Denote $Z_{a,i} \in \{0, 1\}$ as the i-th realization of playing arm a, and $\rho(x; \mu) = \mu^x (1-\mu)^{1-x}$ for $x \in \{0, 1\}$ and $\mu \in [0, 1]$ as the probability for Bernoulli random variables. Then we can define function

$$L(\vec{N}) := \ln \left(\prod_{i=1}^{N_{a_1}} \frac{\rho(Z_{a_1,i}; \mu_1)}{\rho(Z_{a_1,i}; \lambda)} \cdots \prod_{i=1}^{N_{a_{d-1}}} \frac{\rho(Z_{a_{d-1},i}; \mu_1)}{\rho(Z_{a_{d-1},i}; \lambda)} \cdot \prod_{i=1}^{N_{a_d}} \frac{\rho(Z_{a_d,i}; \mu_1)}{\rho(Z_{a_d,i}; \mu_3)} \right). \tag{52}$$

Fix any $\eta \in (0, 3\gamma - 2)$, and define event $\mathcal{L} := \left\{ L(\vec{N}) \leq (1-\eta) \ln T \right\}$. In the following, we will show that

$$\mathbb{P}[\mathcal{N}] = \mathbb{P}\left[N(a_1) < \frac{1-\gamma}{\mathrm{KL}(\mu_1||\lambda)} \ln T \right] \to 0, \text{ as } T \to \infty, \tag{53}$$

by proving that both $\mathbb{P}[\mathcal{N} \cap \mathcal{L}]$ and $\mathbb{P}\left[\mathcal{N} \cap \overline{\mathcal{L}}\right]$ tend to 0.

Step 1: for $\mathbb{P}[\mathcal{N} \cap \mathcal{L}]$. For H'_σ (σ is the greedy sequence), the sequence we play is the greedy sequence only if $a_1, \cdots, a_d$ are chosen simultaneously, thus the total number of playing σ is $N(a_d)$. Furthermore, for any consistent algorithm, for any $\eta \in (0, \gamma)$, we have $\mathbb{E}'[T - N(a_d)] = o(T^\eta)$ under H'_σ. Since in our instance $N(a_1) \geq N(a_d)$ holds, thus $\mathbb{E}'[T - N(a_1)] \leq \mathbb{E}'[T - N(a_d)] = o(T^\eta)$.

$$\begin{aligned}
\mathbb{P}'[\mathcal{N}] =& \mathbb{P}'\left[N(a_1) < \frac{1-\gamma}{\mathrm{KL}(\mu_1 || \lambda)} \ln T\right] \\
=& \mathbb{P}'\left[T - N(a_1) > T - \frac{1-\gamma}{\mathrm{KL}(\mu_1 || \lambda)} \ln T\right] \\
\leq& \frac{\mathbb{E}'[T - N(a_1)]}{T - \frac{1-\gamma}{\mathrm{KL}(\mu_1||\lambda)} \ln T} && \{\text{Markov's inequality}\} \\
=& o(T^{\eta - 1}). && (54)
\end{aligned}$$

Let ρ_x be the probability measure for H^0_σ, then through the change of probability measure, we can get that

$$\begin{aligned}
\mathbb{P}'[\mathcal{N} \cap \mathcal{L}] =& \int_{x \in \mathcal{N} \cap \mathcal{L}} \prod_{i=1}^{N(a_1)} \frac{\rho(Z_{a_1,i}; \lambda)}{\rho(Z_{a_1,i}; \mu_1)} \cdots \prod_{i=1}^{N(a_{d-1})} \frac{\rho(Z_{a_{d-1},i}; \lambda)}{\rho(Z_{a_{d-1},i}; \mu_1)} \cdot \prod_{i=1}^{N(a_d)} \frac{\rho(Z_{a_d,i}; \mu_2 + \frac{1}{4})}{\rho(Z_{a_d,i}; \mu_1)} \, \mathrm{d}\rho_x \\
=& \int_{x \in \mathcal{N} \cap \mathcal{L}} \exp(-L(\vec{N})) \, \mathrm{d}\rho_x && (55) \\
\geq& T^{\eta - 1} \int_{x \in \mathcal{N} \cap \mathcal{L}} \mathrm{d}\rho_x = T^{\eta-1} \, \mathbb{P}[\mathcal{N} \cap \mathcal{L}], && (56)
\end{aligned}$$

where (55) is from (52), and (56) is because $\mathcal{L}$ holds. Thus, we can imply from (56) and (54) that

$$\mathbb{P}[\mathcal{N} \cap \mathcal{L}] \leq T^{1-\eta} \cdot \mathbb{P}'[\mathcal{N} \cap \mathcal{L}] \leq T^{1-\eta} \cdot \mathbb{P}'[\mathcal{N}] = o(1). \tag{57}$$

Step 2: for $\mathbb{P}\left[\mathcal{N} \cap \overline{\mathcal{L}}\right]$. From Equation (52), we know that

$$L(\vec{N}) = \sum_{i=1}^{N(a_1)} \ln\left(\frac{\rho(Z_{a_1,i}; \mu_1)}{\rho(Z_{a_1,i}; \lambda)}\right) + \cdots + \sum_{i=1}^{N(a_{d-1})} \ln\left(\frac{\rho(Z_{a_{d-1},i}; \mu_1)}{\rho(Z_{a_{d-1},i}; \lambda)}\right) + \sum_{i=1}^{N(a_d)} \ln\left(\frac{\rho(Z_{a_d,i}; \mu_2)}{\rho(Z_{a_d,i}; \mu_3)}\right) \tag{58}$$

$$\to (N(a_1) + N(a_2) + \cdots + N(a_{d-1})) \, \mathrm{KL}(\mu_1||\lambda) + N(a_d) \cdot \mathrm{KL}(\mu_1||\mu_3) =: \overline{L}(\vec{N}), \tag{59}$$

as $N(a_1), \cdots, N(a_d)$ tend to be sufficiently large. Due to condition (48) and the definition of $\mathcal{N}$, we have

$$\begin{aligned}
\overline{L}(\vec{N}) \leq& N(1)\left(1 + \frac{1}{W} + \cdots + \frac{1}{W^{d-2}}\right) \mathrm{KL}(\mu_1||\lambda) + N(1) \frac{1}{W^{d-1}} \cdot \mathrm{KL}(\mu_1||\mu_3) \\
\leq& N(1) \frac{1}{1 - \frac{1}{W}} \mathrm{KL}(\mu_1||\lambda) + N(1) \frac{1}{W^{d-1}} \cdot \mathrm{KL}(\mu_1||\mu_3) \\
\leq& \left(\frac{1}{1 - \frac{1}{W}} + \frac{1}{W^{d-1}} \frac{\mathrm{KL}(\mu_1||\mu_3)}{\mathrm{KL}(\mu_1||\lambda)}\right) (1-\gamma) \ln T. && (60)
\end{aligned}$$

Since $2(x-y)^2 \leq \mathrm{KL}(x||y) \leq \frac{(x-y)^2}{y(1-y)}$ for $x, y \in [0, 1)$, $\frac{1}{4} < \mu_1 < \mu_2 < \mu_3 = \frac{3}{4}$, and $\mu_2 < \lambda \leq 1$, it can be implied that

$$\frac{\mathrm{KL}(\mu_1||\mu_3)}{\mathrm{KL}(\mu_1||\lambda)} \leq \frac{(\mu_3 - \mu_1)^2}{\mu_3(1-\mu_3) \cdot 2 \cdot (\lambda - \mu_1)^2} \leq \frac{3}{2(\mu_2 - \mu_1)^2}. \tag{61}$$

Assume that $W \geq 2$ and $W^{d-1} \geq \frac{2}{3(\mu_2 - \mu_1)^2}$, then from (60) and (61) we have $\overline{L}(\vec{N}) \leq 3(1 - \gamma) \ln T$.

Now fix any $\eta \in (0, 3\gamma - 2)$, and it is easy to know that $\overline{L}(\vec{N}) \leq 3(1-\gamma)\ln T < (1-\eta)\ln T$. Recall that $L(\vec{N}) \to \overline{L}(\vec{N})$ from (59) and $\mathcal{N} = \{N(a_1) < \frac{1-\gamma}{\mathrm{KL}(\mu_1||\lambda)} \ln T\} \subsetneqq \bigcup_{j=0}^{\frac{1-\gamma}{\mathrm{KL}(\mu_1||\lambda)} \ln T} \{N(a_1) = j\}$. Then, by the strong law of large numbers, we can get

$$\mathbb{P}\left[\mathcal{N} \cap \overline{\mathcal{L}}\right] = \mathbb{P}\left[N(a_1) < \frac{1-\gamma}{\mathrm{KL}\left(\mu_1||\lambda\right)} \ln T \ \wedge \ L(\vec{N}) > (1-\eta)\ln T\right] \to 0, \text{ as } T \to \infty. \tag{62}$$

Step 3: combine two parts. Because $d \geq \log_W\left(\frac{2W}{3(\mu_2-\mu_1)^2}\right)$, we have that $W^{d-1} \geq \frac{3}{2(\mu_2-\mu_1)^2}$. For any $\gamma \in (\frac{2}{3}, 1)$, it can be derived from (57) and (62) that $\lim_{T\to\infty} \mathbb{P}\left[\mathcal{N}\right] = \lim_{T\to\infty} \mathbb{P}\left[N_T(a_1) < \frac{1-\gamma}{\mathrm{KL}(\mu_1||\lambda)} \ln T\right] = 0$. Then, from (51), we can get

$$\lim_{T\to\infty} \mathbb{P}\left[N(a_1) < \frac{1-\gamma}{(1+\gamma)\,\mathrm{KL}\left(\mu_1||\mu_2\right)} \ln T\right] = 0. \tag{63}$$

Therefore,

$$\lim_{t\to+\infty} \frac{\mathbb{E}\left[N_t(a_1)\right]}{\ln t} \geq \frac{0.39}{\mathrm{KL}\left(\mu_1||\mu_2\right)}. \tag{64}$$

□

We will show that the greedy regret can be as large as $\Omega(\frac{mW \log T}{\Delta^2})$, when (1) the unit gap $(\mu_2 - \mu_1)$ is not exponentially small or (2) m is large enough, which will be specified quantitatively in Theorem E.2.

Theorem E.2. *For instance $\mathcal{P}$ of the prize-collecting problem, for any μ_1, μ_2, μ_3 satisfying that $\frac{1}{4} < \mu_1 < \mu_2 = \frac{1}{2} < \mu_3 = \frac{3}{4}$ and $W \geq 2$, assume that there exists some constant $\xi \in (0,1)$ such that $\xi m \geq \log_W\left(\frac{2W}{3(\mu_2-\mu_1)^2}\right)$ (i.e., $(\mu_2-\mu_1)^2 \geq \frac{2}{3W^{\xi m-1}}$). Denote $\Delta = \mu_2 - \mu_1 > 0$. For any consistent algorithm, as time T tends to $+\infty$, the greedy regret satisfies that*

$$R^{\mathsf{G}}(T) \geq \frac{0.39 \cdot (1-\xi)m(W-1)\Delta_{\min}}{\mathrm{KL}(\mu_1||\mu_2)} \ln T, \tag{65}$$

where $\mathrm{KL}(p||q) = p\ln\frac{p}{q} + (1-p)\ln\frac{1-p}{1-q}$ is Kullback-Leibler divergence for Bernoulli random variables, and $\Delta_{\min} = \Delta + \frac{1}{4}$.

Proof. From the assumption, we know that $\xi m \geq \log_W\left(\frac{2W}{3(\mu_2-\mu_1)^2}\right)$. Fix any $\ell = 1, 2, \cdots, (1-\xi)m$, then it is easy to see that the deviated $d = m - \ell$ arms (denoted as $a_1, a_2, \cdots, a_d$) satisfying $d \geq \log_W\left(\frac{2W}{3(\mu_2-\mu_1)^2}\right)$.

Denote the true greedy sequence as $\sigma^{\mathsf{G}} = \langle G_0, G_1, \cdots, G_m\rangle$. For any sequence σ coinciding with σ^{G} for the prefix $\langle G_0, G_1, \cdots, G_\ell\rangle$, for phase $\ell+1$, we can choose $W-1$ candidates from $\mathcal{N}(G_\ell) \setminus \{g^*_{G_\ell}\}$. According to Lemma E.1, for each candidate, the consistent algorithm needs to play arm a_1 for at least $\frac{0.39}{\mathrm{KL}(\mu_1||\mu_2)} \ln T$ times. Since the penalty is at least $\Delta_{\min} = \Delta + \frac{1}{4}$, therefore we can conclude that

$$R^{\mathsf{G}}(T) \geq \sum_{\ell=1}^{(1-\xi)m} (W-1) \cdot \frac{0.39 \cdot \Delta_{\min}}{\mathrm{KL}(\mu_1||\mu_2)} \ln T = \frac{0.39 \cdot (1-\xi)m(W-1)\Delta_{\min}}{\mathrm{KL}(\mu_1||\mu_2)} \ln T. \tag{66}$$

□

From Theorem E.2 and $\mathrm{KL}(\mu_1||\mu_2) \leq \frac{(\mu_2-\mu_1)^2}{\mu_2(1-\mu_2)} = 4\Delta^2$, it is easy to check that the greedy regret is $\Omega\left(\frac{mW\Delta_{\min}}{\Delta^2}\log T\right)$. Since $\Delta_{\max} = m\Delta + \frac{1}{4}$ and $\Delta_{\min} = \Delta + \frac{1}{4}$ in the setting of Theorem E.2, suppose we further assume that $\Delta = \mu_2 - \mu_1 = o(\frac{1}{m})$ (e.g., $\Delta = \frac{0.01}{m^{1.1}}$), then it indicates that $\Delta_{\max} = \Delta_{\min} + o(1)$. Thus, we can get that the lower bound $R^{\mathsf{G}}(T) = \Omega\left(\frac{mW\Delta_{\max}}{\Delta^2}\log T\right)$, which matches with the upper bound up to a constant factor.

E.2 Lower bound with an exponential term

We further remark that a similar problem instance as in Appendix E.1 can be used to show a regret lower bound with an exponential term: $\Omega\left(\frac{W^m}{(\mu_3-\mu_2)}\log T\right)$.

The intuition is that all the first $m-1$ layers have essentially no difference in reward for any decision sequence, and all reward is on the last layer, i.e., $\mu_3 - \mu_2 \gg \mu_2 - \mu_1 > 0$ and $\mu_2 - \mu_1 = O(\frac{1}{W^m})$, which makes it similar to the classical multi-armed bandit with W^m arms. Therefore, the lower bound is in the order of $\Omega\left(\frac{W^m}{(\mu_3-\mu_2)}\log T\right)$. This means we cannot have $O\left(\frac{\mathrm{poly}(W,m)}{(\mu_3-\mu_2)}\log T\right)$ regret.

F Extensions

F.1 A simple extension of OG-UCB for linear bandits with a matroid constraint to achieve $O(n/\Delta \log T)$ regret and $O(n)$ space

If we know that the problem instance has linear reward function, i.e., $f_t(S) = \sum_{e\in S} f_t(e)$, and the accessible set system $(E, \mathcal{F})$ is restricted to a matroid, we can easily extend OG-UCB and make it behave essentially the same as Algorithm OMM in [23].

The key is to merge those equivalent arms. More formally, we call two arms $a = e|S$ and $a' = e|S'$ *equivalent* if the marginal rewards of both arms follow the same distribution. For these equivalent arms, we merge estimator $\hat{X}(a)$ and $\hat{X}(a')$ (and the counters $N(a)$ and $N(a')$), and it can be achieved simply by using the same memory for a and a'. This applies to the setting of linear reward function [23]: the marginal value of choosing arm $e|S$ or $e|S'$ only depends on element e and is irrelevant to S or S', and thus we merge all such arms $e|S$ and $e|S'$, such that observations of one arm refine the estimation of its equivalent arms. In this case, it can be easily verified that OG-UCB and OMM behave essentially the same (except on some minor manipulation of time counter t' in Line 4 of Algorithm 1).

In this case, we remark that: (1) this extension utilizes the property of linear bandits and matroid constraints; (2) the greedy algorithm will find the optimal solution, and henceforth the greedy regret is also equivalent to the expected cumulative regret (i.e., $R^1(T)$). Therefore, the analysis of OMM in [23] applies to OG-UCB: OG-UCB has regret bound $O(\frac{n}{\Delta}\log T)$ and the memory cost is $O(n)$, where Δ is the minimum unit gap.

F.2 Algorithms for Knapsack Constraints

In this section, we consider the knapsack constraint, which is a special case of our accessible set system $(E, \mathcal{F})$.

For each element $e \in E$, a cost function $\lambda(e) \in \mathbb{R}_{\geq 0}$ is given in advance. Without loss of generality, we assume $\lambda(a) = \lambda(e) \in [1, \infty)$ for each arm $a = e|S \in \mathcal{A}$. Given a budget B for the knapsack problem, the collection of feasible sets are $\mathcal{F} = \{S \subseteq E : \sum_{e\in S}\lambda(e) \leq B\}$. As is described Example A.4, such $(E, \mathcal{F})$ is an accessible set system.

Notice that, for each phase, suppose the offline greedy still maximizes the marginal reward $\overline{f}(a)$ from a set of accessible arms as before, then it is only the special case of the original model described in the main text, therefore the algorithms and results apply without any change.

However, for many problems with knapsack constraints, a natural way of the greedy algorithm is to maximize the marginal reward per unit cost every phase i.e., $\frac{\overline{f}(a)}{\lambda(a)}$ is maximized (instead of maximizing the marginal reward of a set accessible arms). We show that we can slightly modify our previous algorithm to accommodate this setting.

In particular, our algorithm can be slightly modified as follows: in the offline and online problems, $\arg\max$ and $\mathrm{MaxOracle}$ return an arm a such that $\frac{\hat{X}(a)}{\lambda(a)}$ is maximized (replace $\hat{X}(a)$ in the objective of $\arg\max$ and $\mathrm{MaxOracle}$ with $\frac{\hat{X}(a)}{\lambda(a)}$). Then, we can rewrite the *greedy element*:

$$\forall S \in \mathcal{F}, \quad g_S^* := \arg\max_{e \in \mathcal{N}(S)} \frac{\overline{f}(e|S)}{\lambda(e|S)}, \tag{67}$$

together with the *unit gap*:

$$\forall S \in \mathcal{F}, \forall e \in \mathcal{N}(S), \quad \Delta(e|S) := \begin{cases} \frac{\overline{f}(g_S^*|S)}{\lambda(g_S^*|S)} - \frac{\overline{f}(e|S)}{\lambda(e|S)}, & e \neq g_S^* \\ \frac{\overline{f}(e|S)}{\lambda(e|S)} - \max_{e' \in \mathcal{N}_-(S)} \frac{\overline{f}(e'|S)}{\lambda(e'|S)}, & e = g_S^*. \end{cases} \tag{68}$$

The algorithms OG-UCB and OG-LUCB with such modification still work. The rest of the analysis is the same. Therefore, Theorems 3.1, 4.1, 4.2 and 4.3 still apply after rewriting the above definitions.

G Applications

G.1 Top-m Selection with a Consistent Function

In this subsection, we will demonstrate the application of our framework to the problem of Top-m Selection with a consistent reward function.

Generally speaking, the reward of this problem may be non-linear, and the greedy algorithm can output an optimal solution (with approximation ratio $\alpha = 1$).

The decision class. In the top-m selection problem, the set of feasible sets are all subsets of cardinality at most m, hence corresponding to an accessible set system $(E, \mathcal{F})$, where $E = \{1, 2, \cdots, n\}$, and $\mathcal{F} = \{S \subseteq E : |S| \leq m\}$.

Notice that, the arm space is $\mathcal{A} = \{e|S : \forall S, \forall e \in \mathcal{N}(S), |S| \leq m-1\}$, containing $\prod_{i=0}^{m-1}(n-i)$ arms in total.

Definition G.1 (Consistent function [5])**.** A function $\psi : 2^E \to \mathbb{R}$ is *consistent* if for any $T \subset T' \subset E$, and element $x, y \in E \setminus T'$, we have $\psi(T \cup \{x\}) \geq \psi(T \cup \{y\}) \Longrightarrow \psi(T' \cup \{x\}) \geq \psi(T' \cup \{y\})$.

Reward function. Let $\Omega \subseteq [0,1]^E$ be the probability space. For each time $t = 1, 2, \ldots$, the environment draws an i.i.d. samples ω_t from Ω. ω_t can be thought as a vector and $\omega_t(e) \in [0,1]$ for each dimension $e \in E$. For any $\omega_t \in \Omega$, the reward function is a bounded and non-decreasing function $f_t(S) := f(S, \omega_t) \in [L, L+m]$ (fix constant $L \geq 0$) as required in Section 2, and define $\overline{f}(S) = \mathbb{E}[f_t(S)]$ where the expectation is taken over ω_t. We assume that $\overline{f}(\cdot)$ is a *consistent function*.

Offline and online settings. In the offline setting, $\overline{f}(\cdot)$ is provided as a value oracle and we would like to return the subset with the maximum $\overline{f}$ value. In the online setting, the player first plays a decision sequence $\sigma = \langle S_0, S_1, \cdots, S_k \rangle$ with $S_k = S_{k-1} \cup \{s_k\}$ for time t, then observes the semi-bandit feedbacks $\{f_t(S_i)\}_{i=0,\ldots,k}$, and gains a reward of $f_t(S_k)$.

Example G.1. The following functions belong to the consistent function family:

- $\overline{f}(S) = c \cdot e^{\sum_{e \in S} \mathbb{E}[\omega_t(e)]}$ (c is a small constant for normalization);
- $\overline{f}(S) = 1 - \prod_{e \in S} (1 - \mathbb{E}[\omega_t(e)])$;
- $\overline{f}(S) = \frac{1}{2}(|S| + \min_{e \in S} \mathbb{E}[\omega_t(e)])$, with $f_t(\emptyset) = 0$. It is a variant of bottleneck functions, and the use $|S|$ and $\frac{1}{2}$ is for normalization;
- $\overline{f}(S) = \sum_{e \in S} \mathbb{E}[\omega_t(e)]$ (linear function).

Denote the greedy sequence for the top-m selection problem as $\sigma^{\mathsf{G}} = \langle G_0, G_1, \cdots, G_m \rangle$ (m is the length). from Theorem 2 of [5]:

Lemma G.1. *For consistent function $\overline{f}(\cdot)$, G_m is an optimal solution, i.e., $\overline{f}(G_m) = \max_{\forall S\in\mathcal{F}} \overline{f}(S)$ is satisfied.*

We can use our algorithm OG-UCB to solve the online problem by playing the game during the time horizon T. Since the offline greedy solution is also the optimal one (approximation ratio $\alpha = 1$), from Theorem 3.1 and Proposition B.1, we have the following corollary:

Corollary G.2. *For any time T, for the top-m selection problem with a consistent function, our algorithm OG-UCB can achieve a regret*

$$R(T) \le R^{\mathsf{G}}(T) \le \sum_{a\in\Gamma_-(\sigma^{\mathsf{G}})} \left(\frac{6\Delta^*(a)\cdot \ln T}{\Delta(a)^2} + \left(\frac{\pi^2}{3}+1\right)\Delta^*(a) \right), \tag{69}$$

where σ^{G} is the greedy sequence of length m, and $R(T)$ is the expected regret with respect to the optimal solution.

Notice that $\Gamma_-(\sigma^{\mathsf{G}})$ has $\sum_{i=1}^{m}(n-i) = (n-1-\frac{m}{2})m$ arms in the above corollary for the top-m selection.

We claim that other decision classes belonging to a matroid embedding [5] also work through the analysis. (The key is to derive the optimal solution guarantee analogue to Lemma G.1.) To the best of our knowledge, the bound of $O(\log T)$ regret for the online version of the top-m selection problem (or the wider decision classes) with a general consistent function is new.

G.2 Stochastic Online Submodular Maximization

Our algorithms and results can be applied to online submodular maximization. For any time t and for any $S \in \mathcal{F}$, assume the reward function $f_t(S) \in [L, L+m]$ $(L \ge 0)$ is a non-negative monotone submodular function. Denote $\overline{f}(S) := \mathbb{E}[f_t(S)]$, and it is provided for the offline problem. It is obvious that $\overline{f}(\cdot)$ is also non-negative monotone submodular due to the linearity of the expectation.

Let the accessible set system $(E, \mathcal{F})$ be a uniform matroid, in which the maximal feasible set $S \in \mathcal{F}$ is of size m. We adapt the result for offline submodular maximization in [21, 28], with a slight generalization for the greedy (or ϵ-quasi greedy) policy (The proof is almost identical to that in [21], and is omitted here).

Lemma G.3. *Assume that the non-negative monotone submodular function $\overline{f}(\cdot)$ is provided as a value oracle. For any $\epsilon \ge 0$, suppose a decision sequence $\sigma = \langle S_0, S_1, \cdots, S_k\rangle \in \mathcal{F}^{k+1}$ satisfies $S_0 = \emptyset$, $S_i = S_{i-1}\cup\{s_i\}$ for every i, and $\overline{f}(s_i|S_{i-1}) \ge \overline{f}(g^*_{S_{i-1}}|S_{i-1}) - \epsilon$ for every i. Then, for every positive integer ℓ and m,*

$$\overline{f}(S_\ell) \ge \left(1-\left(1-\frac{1}{m}\right)^k\right)\left(\overline{f}(S^*) - m\epsilon\right), \tag{70}$$

where k is the length of the decision sequence and $S^* = \arg\max_{S\in\mathcal{F}} \overline{f}(S)$.

Noting that $1 - x \le e^{-x}$ for all $x \in \mathbb{R}$, we can immediately derive the following two properties from the above lemma.

Lemma G.4. *(1) Let $\sigma^{\mathsf{G}} = \langle S_0, S_1, \cdots, S_m\rangle \in \mathcal{F}^{m+1}$ be the greedy sequence of size m. Then,*

$$\overline{f}(S_m) \ge \left(1-e^{-1}\right)\overline{f}(S^*).$$

Thus, S_m in σ^{G} is an α-approximation solution for $\alpha = 1-e^{-1}$.

(2) Given any small ϵ such that $0 \le \epsilon \le \frac{L}{m}$, let $\sigma^{\mathsf{Q}} = \langle S_0, S_1, \cdots, S_m\rangle \in \mathcal{F}^{m+1}$ be the minimum of ϵ-quasi greedy sequence. Then,

$$\overline{f}(S_m) \ge \left(1-e^{-1}-\frac{m\epsilon}{L}\right)\overline{f}(S^*).$$

Thus, S_m in σ^{Q} is an α-approximation solution for $\alpha = 1-e^{-1}-\frac{m\epsilon}{L}$.

Now we can use Algorithm OG-UCB and OG-LUCB to solve the online version of the submodular maximization problem. From Propositions B.1 and B.2, we know that $R^\alpha(T) \leq R^{\mathsf{G}}(T)$ and $R^\alpha(T) \leq R^{\mathsf{Q}}(T)$ for the respective α-approximation solutions. Then, from Theorem 3.1 and Theorem D.7, and we have the following α-approximation regret:

Corollary G.5. *For any time horizon T, for the online submodular maximization problem, our algorithm OG-UCB can achieve the following α-approximation regret:*

$$R^\alpha(T) \leq R^{\mathsf{G}}(T) \leq \sum_{a \in \Gamma_-(\sigma^{\mathsf{G}})} \left(\frac{6\Delta^*(a) \cdot \ln T}{\Delta(a)^2} + \left(\frac{\pi^2}{3} + 1 \right) \Delta^*(a) \right), \tag{71}$$

where σ^{G} is the greedy sequence, and $\alpha = 1 - e^{-1}$ is the approximation ratio of the offline problem.

Corollary G.6. *For the online submodular maximization problem, suppose we run Algorithm OG-LUCB-R with some small ϵ such that $0 \leq \epsilon \leq \frac{c}{m}$, and let $\alpha = 1 - e^{-1} - \frac{m\epsilon}{L}$ be the approximation ratio of the offline problem.. Use ϕ_i, τ_i, $\sigma^{(\ell)}$, K, $R^{\mathsf{Q},\sigma^{(\ell)}}(\cdot)$, c_ℓ and b_ℓ defined in Theorem D.7. Our algorithm OG-LUCB-R can achieve an α-approximation regret as follows:*

$$R^\alpha(T) \leq R^{\mathsf{Q}}(T) \leq \sum_{\ell=1}^{K} R^{\mathsf{Q},\sigma^{(\ell)}}(\phi_\ell). \tag{72}$$

Furthermore, we can get

$$R^\alpha(T) \leq R^{\mathsf{Q}}(T) \leq 4 \cdot \left(\max_{\ell=1,\cdots,K} c_\ell \right) \cdot \ln(T) + \left(\max_{\ell=1,\cdots,K} b_\ell \right) \cdot \log_2 (2 \ln T). \tag{73}$$

Besides, the exploration time for Algorithm $\mathsf{OG\text{-}LUCB}_{\epsilon,\delta}$ for any input of (ϵ, δ) is the same as Theorem 4.1; and letting $\delta = \frac{1}{T}$ implies a corollary for the α-approximation regret R^α ($R^\alpha \leq R^{\mathsf{Q}}$ and $\alpha = 1 - e^{-1}$), corresponding to Theorem D.5, both of which will not be restated here.

The above results for stochastic online greedy, ensuring the $O(\log T)$ bounds, are complementary to the $O(\sqrt{T})$ α-approximation regret bound presented in [31], which focuses on the online submodular maximization in the adversarial setting.

G.3 Online Influence Maximization and Probabilistic Set Cover in a Unified Model

From the above Appendix G.2, we have shown that our algorithms work without assuming the detailed form of the reward function. In light of this, our model can be applied to unify the online *Influence Maximization (IM)* and *Probabilistic Set Cover (PMC)* problems [19] across different models. (PMC can be viewed as two-layers influence maximization on the independent cascade model.)

Influence maximization problem [19] has been widely studied in the viral marketing. In general, given social graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ representing nodes and edges respectively, the goal is to choose a m-nodes set $S \subseteq \mathcal{V}$ as *seeds*, so that starting from the seed nodes, at many nodes as possible are influenced (or covered) via the word-of-mouth effect. Many models, such as the *independent cascade (IC)* model, the *linear threshold (LT)* model [19] and the *continuous-time independent cascade (CIC)* model [30], are proposed to capture different natures of users' behaviors. The common part of those models (IC, LT, CIC, etc.) is that they assume that the coverage function $f : 2^{\mathcal{V}} \times \Omega \to \mathbb{R}$ (and we use $f_t(S) = f_t(S, \omega_t)$ for convenience, where ω_t is the randomness of the environment at time t), and focus on maximizing the expected coverage $\overline{f}(S) := \mathbb{E}[f_t(S)]$ called *influence spread*, which is a non-negative monotone submodular function. Note that due to the #P-hardness of computing $\overline{f}(\cdot)$ [10] in order to find the optimal solution, Monte Carlo simulations on $f_t(\cdot)$ are usually carried out to estimate $\overline{f}(\cdot)$. The difference of models lies in the assumption of the environmental randomness Ω: in IC model, it comes from independent Bernoulli random variables assigned on edges; CIC model parametrizes edges with time-variant distributions, and a constant total cutting time; and in LT model, the randomness are from the thresholds of nodes. The empirical study in [13] shows that the model misspecification may lead to one optimal solution for one model yielding a bad performance in another.

In the online version of influence maximization, we need to choose the seeds set S over time to maximize the overall performance against the optimal S^* in hindsight. [11] shows how to model the online influence maximization for IC model as a combinatorial multi-arm bandit problem. If the IC model is assumed, the randomness of edges can be modelled as base arms and the reward is a function represented by the expected values of those arms. However, in our framework, we do not need the knowledge of the detailed diffusion model or the exact form of the reward function. For different influence maximization models, we only access the value of $f_t(\cdot)$ without assuming the exact form $f_t(\cdot)$ or $\overline{f}(\cdot)$. The application of our framework, utilizing value oracle $f_t(\cdot)$, can unify the above models of online influence maximization as well as avoid the risk of model misspecification. As is illustrated in Appendix G.2, our algorithms still apply here and the same results hold.

H Empirical Evaluation for the Lower Bound on the Prize-Collecting Problem

In this section, we carry out an empirical evaluation for the prize-collecting problem to validate Algorithms OG-UCB. We use the problem instance $\mathcal{P}$ as described in Appendix E.

Setup. We evaluate the performance over the following combinations: (1) Select one out of $W \in \{10, 20, 30\}$ elements, for each of $m \in \{4, 6, 8\}$ bandits; (2) The minimum unit gap Δ is chosen from $\{0.2, 0.1\}$ (that is, $\mu_1 \in \{0.3, 0.4\}$ and $\mu_2 = 0.5$ with $\Delta = \mu_2 - \mu_1$), and $\mu_3 = 0.75$.

For each case, we set the same total horizon $T = 10^6$. The lower bound (LB) is estimated by the right-hand side of (65) in Theorem E.2.

The experiment is repeated for 20 times, and we calculate the average of the regret together with their standard deviation.

Analysis. Table 2 illustrates the result of Algorithms OG-UCB. The result contains three columns of values: The left column is regrets of OG-UCB with their standard deviation in absolute values ($\times 10^4$), the middle column is the lower bound estimated in absolute values ($\times 10^4$), and the right column (surrounded by parentheses) is the ratio between regrets in practice and LB.

First, observing the ratio between regrets and LB, we know that it is about 20 times. The ratio does not increase with W or m. Therefore, it indicates that our upper bound matches with the lower bound, and they are tight up to a constant factor.

Second, comparing the regret of algorithms with the change of W or m, we see that the regret increase linearly with W or m. It matches with the regret $\Theta(\frac{Wm\Delta_{\max}}{\Delta^2} \log T)$ derived from our theoretical analysis. When Δ is changed from 0.2 to 0.1, the regret is also about 4 times of the original. (It is less than 4 times because $\Delta_{\max}$ shrinks meanwhile.)

Table 2: Experiment result of Algorithms OG-UCB for the prize-collecting problem ($\mathcal{P}$).

W	m	Δ	OG-UCB ($\times 10^4$)	LB ($\times 10^4$)	(OG-UCB/LB)
10	4	0.20	1.17 ± 0.06	0.047	(24.8)
10	4	0.10	2.80 ± 0.12	0.099	(28.3)
10	6	0.20	2.40 ± 0.07	0.100	(23.9)
10	6	0.10	5.56 ± 0.19	0.268	(20.8)
10	8	0.20	3.88 ± 0.14	0.153	(25.3)
10	8	0.10	9.00 ± 0.26	0.436	(20.6)
20	4	0.20	2.45 ± 0.05	0.115	(21.3)
20	4	0.10	6.01 ± 0.16	0.284	(21.1)
20	6	0.20	4.99 ± 0.12	0.227	(21.9)
20	6	0.10	11.54 ± 0.32	0.640	(18.0)
20	8	0.20	8.24 ± 0.17	0.339	(24.3)
20	8	0.10	18.55 ± 0.34	0.996	(18.6)
30	4	0.20	3.78 ± 0.08	0.186	(20.4)
30	4	0.10	9.04 ± 0.25	0.479	(18.9)
30	6	0.20	7.59 ± 0.10	0.357	(21.3)
30	6	0.10	17.55 ± 0.40	1.023	(17.2)
30	8	0.20	12.61 ± 0.17	0.528	(23.9)
30	8	0.10	28.23 ± 0.38	1.566	(18.0)