The Power of Choice

Towards a Logic for Polynomial Time

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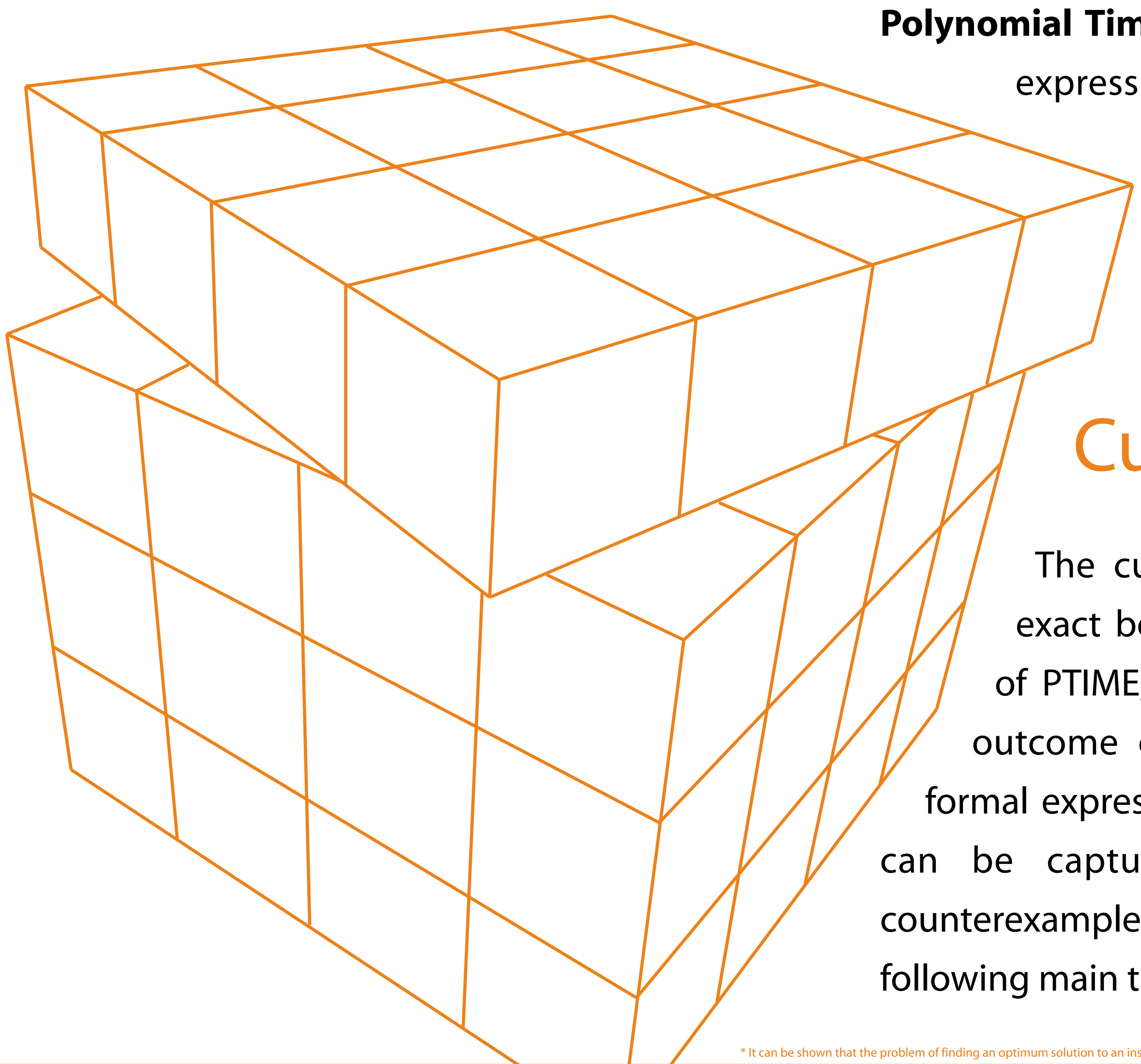
Measuring Complexity with Logic

Computational complexity theory studies the scalability of algorithms in terms of time, memory, number of processors, and other measurable quantities of the computational model. From this study we can separate computational problems into different complexity classes, such as PTIME, the collection of all decision problems that can be solved by deterministic polynomial-time algorithms.

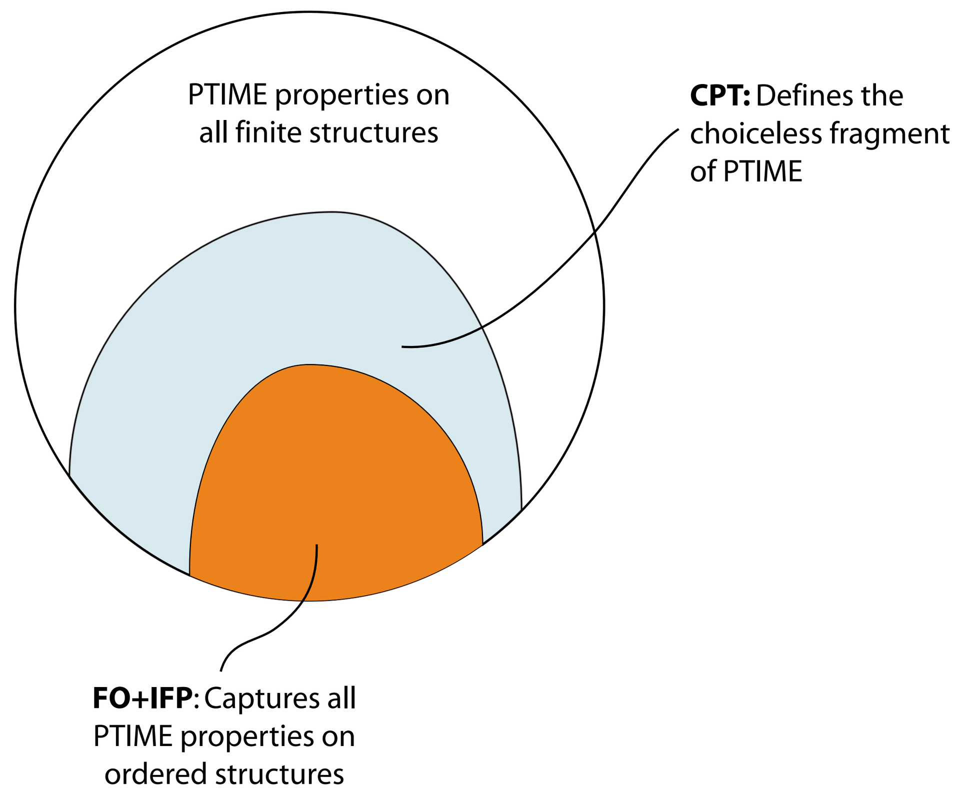
In recent years, researchers have considered an alternative way of studying computational complexity by using techniques from logic and finite model theory. In **descriptive complexity** we consider the richness of the least logic that is needed to precisely *describe* a problem, rather than asking how difficult it is to *compute* a problem. These two notions of complexity—the efficiency of computation *vs* the richness of a logic—turn out, in many cases, to be equivalent.

The first major discovery in this area was made by Fagin, who showed that the class NP, which consists of decision problems that can be solved by non-deterministic polynomial-time algorithms, is exactly the set of problems that can be described in existential second-order logic. While many other compexity classes have been described in logical terms in this manner, it still remains unknown how we can define a logic which accurately captures all of PTIME. This problem remains one of the biggest challenges in descriptive complexity.



The Challenge: Logic for Polynomial Time

The first result towards a PTIME logic was made by Immerman and Vardi, who independently showed that first-order logic with an additional fixed-point operator (FO+IFP) describes exactly all the polynomial-time properties on the class of ordered structures, which are structures that are equipped with a linear order of their elements. Unfortunately, this logic is not a candidate for capturing PTIME on all structures, as it fails to express some very simple queries in the absence of order.



One way to investigate the logical definability of PTIME on unordered structures is to consider how much we can express in a logic which forbids arbitrary *choice*. Choice and order can be seen as two sides of the same coin: in the presence of an order, one can always simulate choice by picking the least element in a relation, say, and with an unrestricted choice one can always impose an order on a set by repeatedly choosing one element after another.

Blass, Gurevich and Shelah follow this line of investigation and define the logic **Choiceless Polynomial Time** (CPT). This logic is strictly more expressive than first-order logic with a fixed-point operator and it is an open question whether some of its variants can capture all of polynomial time.

Current Research

The current research aims to establish exact bounds on the choiceless fragment of PTIME, as defined by the logic CPT. The outcome of this research will either be a formal expressibility proof, outlining how much can be captured by CPT, or a separating counterexample. Our work can be divided into the following main tasks:

Descriptive Complexity and the "**PTIME** *vs* **NP**" Question

The question whether the complexity classes PTIME and NP are equal is considered one of the most important open problems in theoretical computer science. Descriptive complexity offers a novel approach to this problem.

The class NP has been shown to be equivalent to the class of decision problems which can be described in existential second-order logic (Fagin's theorem). On ordered structures we also know that first-order logic with a fixed-point recursion mechanism describes exactly the polynomial-time computable properties. Should someone find a logic which could capture PTIME without this additional order restriction, then the "PTIME *vs* NP" question would be reduced to a purely logical problem—separating two logics over the class of all finite structures.

A. Systematically study problems that lie on the boundary of other established fragments of PTIME. This work, currently in progress, will either identify a weakness of the CPT logic or give further evidence of its strength.

B. Although technically a "logic", CPT is defined in terms of a machine model which makes it difficult to apply conventional tools from finite model theory. This machine model also has no built-in guarantee that CPT statements will be evaluated within polynomial-time bounds, which means that such bounds have to be applied externally. One of our main objectives is to develop a more "logical" form of the CPT model, which will address these shortcomings.

Future Work

Our next step is to investigate the relationship between CPT and another candidate logic that has been proposed for capturing PTIME over unordered structures. This logic is obtained by adding symmetric, non-deterministic choice to FO+IFP, which yields a semi-deterministic logic which can define queries up to isomorphism. Proving that CPT includes the expressive power of this special non-deterministic choice would give us a very interesting result—effectively showing that our "choiceless" model can make some limited choice after all!

This research is currently at an early stage and is expected to finish in 2009.

* It can be shown that the problem of finding an optimum solution to an instance of the $N \times N \times N$ Rubik's cube belongs to the complexity class PSPACE. An optimal solution specifies the least number of rotations required to return the cube to the position where each face has the same colour, for any possible scrambling of the cube.