Local Overlaps In Special Unfoldings Of Convex Polyhedra

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Abstract

We define a notion of local overlaps in polyhedron unfoldings. We use this concept to construct convex polyhedra for which certain classes of edge unfoldings contain overlaps, thereby negatively resolving some open conjectures. In particular, we construct a convex polyhedron for which every shortest path unfolding contains an overlap. We also present a convex polyhedron for which every steepest edge unfolding contains an overlap. We conclude by analyzing a broad class of unfoldings and again find a convex polyhedron for which they all contain overlaps.

1 Introduction

Polyhedron unfolding is a well-studied operation in computational geometry. In an edge unfolding, one unfolds a polyhedron by cutting its surface along edges and flattening it into the plane. A common problem is determining whether or not the resulting surface is simple, meaning that it lies in the plane without overlap. Additionally, one may ask whether *any* unfolding of a given polyhedron is simple.

Shephard conjectured that every convex polyhedron has a simple edge unfolding [7]. It is generally believed that this conjecture is true, but a resolution has proved elusive despite many decades of research. In attempts to resolve Shephard's Conjecture, some researchers have defined classes of unfoldings that are proposed to be simple for convex polyhedra [3, 6].

In this paper we consider a particular type of overlap, which we call a 2-local overlap. This class of overlaps is designed for simplicity of analysis, as their occurrence depends on a local configuration of cuts in a polyhedron. We exploit this property to develop conditions on a convex polyhedron and cut tree that guarantee that a 2-local overlap will occur in the corresponding unfolding. We then use this result to construct convex polyhedra for which every unfolding in given classes contains an overlap, negatively resolving some open conjectures.

In Section 2 we provide formal definitions used throughout the paper. Section 3 contains our analysis of 2-local overlaps and the conditions under which a 2-local overlap will occur. The subsequent sections apply these results to different classes of cut trees: shortest path cut trees in Section 4, steepest edge cut trees in Section 5, and normal order cut trees in Section 6. In Section 7 we provide some concluding remarks and discuss possible avenues of future research.

2 Definitions

The *exterior angle* of a polygon at vertex w is the angle formed externally by the two edges incident with w. The *interior angle* at w is the angle facing the interior. An interior angle of a face of a polyhedron is also called a *face angle*. If v is a vertex of a polyhedron, the *total face angle* at v is the sum of all face angles at v and the *curvature* at v is 2π minus the total face angle at v.

An *edge unfolding* of a polyhedron is obtained by cutting a subset of its edges and unfolding the resulting surface into a connected planar piece. The edges

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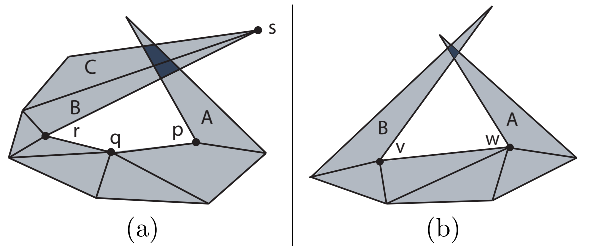

Figure 1: Examples of k-local overlaps for (a) $k = 3, 4$ and (b) $k = 2$.

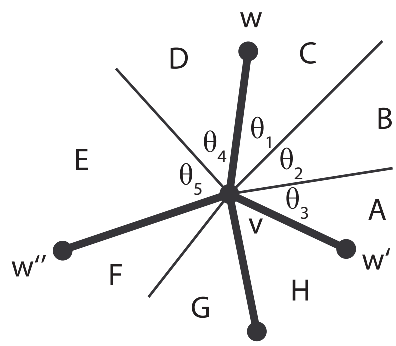

Figure 2: Unfolding angles at vertex v. The unfolding groups at v are ABC, DE, FG, and H. The unfolding angle bounded by (v, w) and (v, w') is $\theta_1 + \theta_2 + \theta_3$. The unfolding angle bounded by (v, w) and (v, w'') is $\theta_4 + \theta_5$.

that are cut in this process will form a spanning tree of the vertices called a *cut tree*. The dual of the cut tree is the *adjacency tree*, in which two faces are connected if their common edge is not cut. Note that a polyhedron can have multiple unfoldings, depending on which edges are cut. We say that the edge unfolding resulting from cutting along cut tree C is *the unfolding associated with* C. A *simple* edge unfolding is one that lies in the plane without overlap.

Suppose a polyhedron unfolding has an overlap between two faces, f_1 and f_2. This overlap is called *k-local* if there are at most k vertices in the shortest path of the unfolding that starts with a vertex incident with f_1 and ends with a vertex incident to f_2. In Figure 1(a) the overlap between faces A and B is 3-local, corresponding to points p, q, and r. The overlap between faces A and C is 4-local, as it involves point s as well. Figure 1(b) shows an example of a 2-local overlap. Note that a 2-local overlap occurs precisely when there is an edge (v, w) in the unfolding such that the overlapping faces are incident with vertices v and w, respectively.

Let C be a cut tree and consider the faces incident with a vertex v. Let the images of v in the unfolding be $v_1, \ldots, v_k$. Then all of the faces incident with a given v_i in the unfolding form an *unfolding group* or *component* of v. No face can belong to more than one unfolding group of v; such a face would have to be incident with v at two points along its boundary.

There is a relationship between unfolding groups and the cut tree C. Two faces f_1 and f_2 are in the same unfolding group precisely when one can traverse faces incident with v from f_1 to f_2 around v (either clockwise or counterclockwise) without crossing an edge in C. In other words, the edges in C split the faces incident with v into the unfolding groups. This implies that the number of unfolding groups at v is precisely the degree of v in C. See Figure 2.

The sum of the face angles at v over all faces in an unfolding group is called an *unfolding angle* at v. The unfolding angles at v are precisely the interior angles of $v_1, \ldots, v_k$ in the unfolding. This is because the interior angle at some v_i is simply the sum of all face angles at v_i, which is the same as the face angles at v for all faces in the unfolding group corresponding to v_i.

Finally, if e is a cut edge incident with v, we say that an unfolding group is *bounded by* e if a face in the group is incident with e. The unfolding angle of such a group is referred to as an *unfolding angle bounded by* e. See Figure 2.

3 Characterizing 2-Local Overlaps

We shall now develop conditions for cut trees on convex polyhedra that will result in 2-local overlaps. The core idea is illustrated in Figure 4. In that figure, the face incident with w fits tightly into the space around vertex v. Thus, if the curvature at w is small, the face incident with w cannot "swing out" enough to clear the faces incident with v. Note that this unfolding pattern is similar to Schlickenrieder's unfolding of hanging facets [6] and to the unfolding of polyhedral bands studied by Aloupis et al. [1].

We begin the formal proof by providing a set of conditions on an unfolding that implies the presence of a 2-local overlap.

Lemma 1 *Suppose P' is an unfolding of a convex polyhedron. Let e_1, e_2, and e_3 be incident edges on the boundary of P', where e_1 and e_2 have common vertex v and e_2 and e_3 have common vertex w. Further suppose that $|e_3| = |e_2|$. Let ϕ be the exterior angle at v, and let θ be the exterior angle at w. If*

1. $\theta + 2\phi < \pi$, *and*
2. $|e_1| \geq |e_2| \frac{\sin\theta}{\sin(\pi-\theta-\phi)}$

then P' will contain a 2-local overlap.

Proof. See Figure 3 for an illustration of the statement of this lemma.

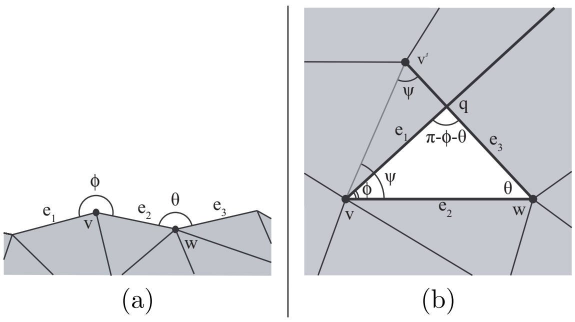

Figure 3: Unfoldings in Lemma 1. Shaded areas represent interiors of faces. (a) The configuration of edges, vertices, and angles in the statement. (b) A 2-local overlap, showing derivation of the edge length condition. Note that the line drawn from v to v' is not an edge; it is meant to illustrate angle ψ.

Note first that $\theta \leq \pi$ and $\phi \leq \frac{\pi}{2}$ by the first condition in the claim.

Let v' be the vertex besides w incident with e_3. Consider the isosceles triangle formed by v, v', and w. This triangle has angle θ at w, and angle $\psi := \frac{\pi-\theta}{2}$ at v and v'. But we know that $\theta + 2\phi \leq \pi$, so $\phi \leq \frac{\pi-\theta}{2} = \psi$. Thus edge e_1 will intersect e_3, assuming e_1 is sufficiently long.

We now determine the required length of e_1. Extend edge e_1 from v until it intersects e_3. Call that point of intersection q. Consider now the triangle formed by v, w, and q. The angle at q will be $\pi-\theta-\phi$. See Figure 3(b). Then, by the sine rule (and since $|e_2| = |e_3|$), we have that

$$\frac{|q-v|}{\sin\theta} = \frac{|e_3|}{\sin(\pi-\theta-\phi)}.$$

We conclude that e_1 will contain point q, and hence intersect e_3, if

$$|e_1| \geq |q-v| = |e_3| \frac{\sin\theta}{\sin(\pi-\theta-\phi)}$$

as required. □

We are now ready to prove the main result of this section. The following Lemma presents conditions on a cut tree and convex polyhedron that imply a 2-local overlap will occur in the corresponding unfolding.

Lemma 2 *Let P be a convex polyhedron with cut tree C. Suppose the following conditions hold:*

1. *$w \in V(P)$ has degree 1 in C, and is adjacent to $v \in V(P)$ in C.*
2. *There is an unfolding angle ϕ_0 at v bounded by (v, w) with $\phi_0 > \frac{3\pi}{2}$.*
3. *There is a value $\gamma > 0$ such that $\frac{|e1|}{|e2|} > \gamma$ for any two edges e_1 and e_2 incident with v.*

Then there exists an angle θ_0 that depends on γ and ϕ_0 such that the unfolding implied by C will contain a 2-local overlap if the curvature at w is less than θ_0.

Proof. See Figure 4 for an illustration of the statement of this lemma.

Let P' be the unfolding of P associated with cut tree C, illustrated in Figure 4(b). We shall show that

P' satisfies the conditions of Lemma 1. Note that the two edges incident with w have equal length, as they are both images of the same edge in C. Let ϕ be the exterior angle at v. Then $\phi = 2\pi - \phi_0$, as ϕ_0 is the interior angle at v. Let θ be the exterior angle at w in the unfolding P'. Then θ is also the curvature of w in P, and hence $\theta < \theta_0$. Let the two edges on the boundary of P' incident with v be e_1 and e_2, where e_2 is incident with w.

Then for all $0 < \theta < \pi - 2\phi$ we have that

$$\theta + 2\phi < \pi. \tag{1}$$

Further, if $\sin\theta < \gamma \sin(\pi - \theta - \phi)$, then

$$|e'| > |e_1| \frac{\sin\theta}{\sin(\pi - \theta - \phi)}.$$

But as $\theta \to 0$, we have $\sin\theta \to 0$ and $\sin(\pi-\theta-\phi) \to \sin(\pi - \theta) > 0$. We conclude that there exists some $\theta_1 > 0$ such that, for all $0 < \theta < \theta_1$,

$$\sin\theta < \gamma \sin(\pi - \theta - \phi) < \frac{|e_1|}{|e_2|} \sin(\pi - \theta - \phi). \tag{2}$$

Take $\theta_0 = \min\{\pi - 2\phi, \theta_1\}$. If the curvature at w is less than θ_0 then the conditions of Lemma 1 are satisfied by Equations 1 and 2, so P' will indeed contain a 2-local overlap.

□

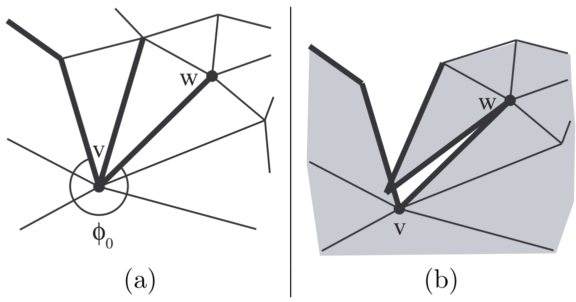

Figure 4: The conditions of Lemma 2. (a) Part of the surface of a polyhedron with cut edges in bold and $\phi_0 > \frac{3\pi}{2}$. (b) The resulting 2-local overlap.

3.1 Discussion

An important feature of the conditions in Lemma 2 is that they are all *local*. That is, Lemma 2 does not depend on any features of the polyhedron or the cut tree beyond the faces and edges incident with vertices v and w. Indeed, this locality is the primary motivation for our definition of k-local unfoldings.

One application of Lemma 2 is the construction of particular convex polyhedra and cut trees with non-simple unfoldings. Informally speaking, if a polyhedron and cut tree can be formed in such a way that some portion of the configuration looks like Figure 4(a), then we can conclude that a 2-local overlap will occur given that curvatures can be made arbitrarily small. The remainder of this paper explores examples of this process.

4 Shortest Path Unfoldings

Lemma 2 provides a tool for constructing convex polyhedra and cut trees that will generate overlaps. We apply Lemma 2 to a particular class of unfoldings. Given a polyhedron P and a vertex $v \in V(P)$, the *shortest path tree at* v, $SPT(v)$, is the tree formed by taking the union of the shortest paths from each vertex $w \in V(P)$ to v along the edges of P.

Fukuda made the following conjecture [3]:

Conjecture 1 (Fukuda) *For every convex polyhedron P and every vertex $v \in V(P)$, the cut tree $SPT(v)$ forms a simple unfolding of P.*

It should be noted that Schlickenrieder has already found an example of a convex polyhedron that disproves this conjecture [6]. We shall construct a different counterexample as an introduction to the methodology used in subsequent sections.

Theorem 3 *There exists a convex polyhedron P with vertex $v \in V(P)$ such that the unfolding corresponding to cut tree $SPT(v)$ contains a 2-local overlap.*

Proof. Consider the graph shown in Figure 5(b). The tree $SPT(b)$ is illustrated in that figure. Place the graph in the plane, taking $|(a, b)| = 1$. We can

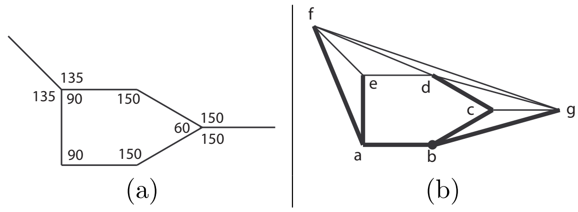

Figure 5: The planar figure used to disprove Conjecture 1. (a) The underlying structure. All line segments are of length 1 and angles are shown in degrees. (b) The completed figure. The bold line segments form $SPT(b)$.

turn this graph into a convex polyhedron by raising vertices c, d, and e off the plane, say by a maximum distance α. This forms a convex terrain, to which we add a bottom face to form a convex polyhedron. Call this polyhedron $P(\alpha)$.

Note that if α is sufficiently small, then the resulting polyhedron has edge lengths and face angles arbitrarily close to that of the planar figure. In particular, there exists some $\epsilon_1 > 0$ such that for all $0 < \alpha < \epsilon_1$, $SPT(b)$ is as shown in Figure 5(b), and faces (b, c, g) and (c, d, g) together form a component with angle greater than $\frac{3\pi}{2}$ at c. Let γ be the minimum value of $\frac{|e_1|}{|e_2|}$ for any two edges e_1 and e_2 incident with b in any $P(\alpha)$, $0 < \alpha < \epsilon_1$. Note $\gamma > 0$.

Then by Lemma 2, there exists some $\theta_0 > 0$ such that the unfolding of $P(\alpha)$ by cutting along $SPT(b)$ will contain a 2-local overlap if the curvature at d is less than θ_0. This same value of θ_0 applies to all $P(\alpha)$, $0 < \alpha < \epsilon_1$. But note that the curvature at d approaches 0 as $\alpha \to 0$. Thus there exists some $\epsilon_2 > 0$ such that $\alpha < \epsilon_2$ implies that the curvature at d is less than θ_0 in $P(\alpha)$.

We conclude that if $0 < \alpha < \min\{\epsilon_1, \epsilon_2\}$ then $P(\alpha)$ contains a 2-local overlap when it is cut along $SPT(b)$. $\square$

5 Steepest Edge Unfoldings

We now consider a more complex class of unfoldings, the steepest edge unfoldings. This class of unfoldings

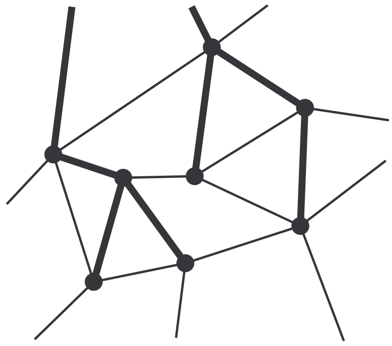

Figure 6: Illustration of steepest edges, where ζ is directed toward the top of the page.

was proposed by Schlickenrieder [6]. As in the previous section, we shall construct a convex polyhedron for which every such unfolding contains an overlap.

5.1 Definition

Let P be a convex polyhedron. Choose a direction vector ζ. Without loss of generality $\zeta = (0, 0, 1)$ by reorienting space. Then for every vertex v in P, let the *steepest edge for* v be (v, w) such that $\frac{w-v}{|w-v|}$ has maximal z-coordinate. That is, the steepest edge is the edge directed most toward ζ from v. The *steepest edge cut tree* contains the steepest edges of all vertices, except the vertex with maximal z-coordinate. A *steepest edge unfolding* is formed by cutting along a steepest edge cut tree. See Figure 6.

Conjecture 2 (Schlickenrieder) *Every convex polyhedron P has a simple steepest edge unfolding.*

This conjecture was motivated by empirical tests, where Schlickenrieder found that a convex polyhedron would unfold without overlap with probability 0.93 when ζ was chosen at random [6]. Nevertheless, we shall construct a polyhedron for which every steepest edge cut tree generates an unfolding with a 2-local overlap.

5.2 Outline

We begin by constructing a convex terrain for which the steepest edge cut tree generates an overlap when ζ lies in some open set. Furthermore, the size of this set is independent of scaling, translation, and rotation of the terrain. We then construct a convex polyhedron by gluing together many copies of this terrain in various orientations. The result will be that for every possible choice of ζ, there is a copy of our terrain that contains an overlap in the corresponding steepest edge unfolding.

5.3 The Terrain

Consider the planar graph M_1 illustrated in Figure 7(a). As with the graph in Section 4, we can convert M_1 into a convex terrain by raising the interior vertices a and b. Given parameter $\alpha \geq 0$, we denote by $M_1(\alpha)$ the convex terrain formed by raising the vertices a and b to a height of at most α in such a way that the resulting terrain remains convex. In particular, raising a to a height of $\frac{\alpha}{2}$ and b to a height of α will result in a convex terrain for all $\alpha > 0$. Also, note that as $\alpha \to 0$, the curvatures at a and b become arbitrarily small.

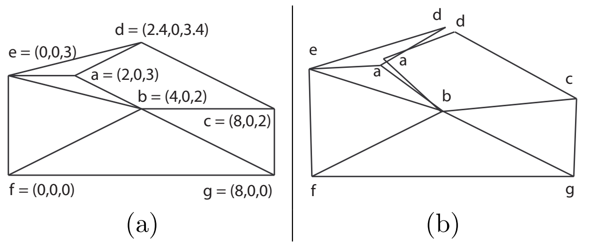

Figure 7: (a) The planar graph M_1. (b) The steepest edge unfolding of $M_1(\alpha)$ for small α and direction vector $\zeta = (0, 0, 1)$.

Lemma 4 *There exists $\phi > 0$ such that, for any sufficiently small $\alpha > 0$, if $M_1(\alpha)$ forms part of convex polyhedron P and ζ is a unit vector within an angle of ϕ from vector $\frac{e-f}{|e-f|}$, the steepest edge unfolding of P with direction ζ will contain a 2-local overlap.*

Proof. See Figure 7(b) for an illustration of the steepest edge unfolding of $M_1(\alpha)$ when $\zeta = \frac{e-f}{|e-f|} = (0, 0, 1)$. Note that vertex b has degree 1 in the unfolding. Also, since $\angle dab < \frac{\pi}{2}$ in M_1, there is an unfolding angle bounded by (a, b) that is greater than $\frac{3\pi}{2}$ for sufficiently small α. The overlap illustrated is implied by Lemma 2 when α is sufficiently small.

Note that the steepest edge cut tree of $M_1(0)$ remains the same given small perturbations of the terrain. In particular, there exists $\phi > 0$ and $\alpha > 0$ such that, for all $0 < \alpha_0 < \alpha$ and $0 \leq \phi_0 < \phi$, the steepest edge unfolding of $M_1(\alpha_0)$ will be as illustrated in Figure 7(b) when ζ is adjusted by an angle of ϕ_0. Thus, for any sufficiently small α, the 2-local overlap described above will occur for $M_1(\alpha)$ if ζ is within an angle of ϕ from $\frac{e-f}{|e-f|}$. Note that our choice of α is independent of our choice of ϕ, provided α is sufficiently small.

Suppose polyhedron P contains an embedded copy of the terrain $M_1(\alpha)$. Consider the vector $\frac{e-f}{|e-f|}$ in this embedded copy of $M_1(\alpha)$. Suppose ζ is within angle ϕ of $\frac{e-f}{|e-f|}$, and consider the steepest edge cut tree of P with direction ζ. Then the unfolding of P restricted to $M_1(\alpha)$ is precisely the unfolding shown in Figure 7(b), as the cut tree over $M_1(\alpha)$ remains the same. Thus the unfolding of P contains a 2-local overlap.

Note that we described ζ with respect to e and f, rather than $(0, 0, 1)$, to make clear the independence of ϕ from any rotation and scale of the terrain that may occur in the course of embedding it into a convex polyhedron. $\square$

5.4 The Polyhedron

We now construct our final polyhedron. We begin with the following technical lemma.

Lemma 5 *Given $\phi > 0$, there exists a convex polyhedron P and a finite set of direction vectors $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ such that*

1. *for any direction vector ζ, there is a ζ_i such that the angle between ζ and ζ_i is less than ϕ, and*

2. *for each ζ_i there is a distinct face f_i of P such that the normal at f_i is perpendicular to ζ_i.*

Proof. Note that a direction vector is equivalent to a point on the unit sphere S. Given direction vector ζ, let $D(\zeta)$ be the set of all direction vectors within an angle of ϕ from ζ. Then $D(\zeta)$ is an open set, and

$$\bigcup_{\zeta \in S} D(\zeta) = S.$$

Since S is a compact set, it follows that there is some finite number of sets $\{D(\zeta_i)\}_{i=1}^n$ such that

$$\bigcup_{i=1}^{n} D(\zeta_i) = S.$$

Then for any $\zeta \in S$, ζ must lie in some $D(\zeta_i)$, and hence ζ is within angle ϕ of ζ_i. Thus $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ is our desired set of direction vectors.

We shall now construct our polyhedron P using the well-known Minkowski Existence Theorem [2]. Choose direction vectors $\gamma_1, \ldots, \gamma_n$ such that γ_i is perpendicular to ζ_i and no two γ_i and γ_j are parallel. Let $A = \{\gamma_1, \ldots, \gamma_n\} \cup \{(0,0,1), (0,1,0), (1,0,0)\}$ be a set of direction vectors. Note that A spans $\mathbf{R}^3$. Define vector Γ by

$$\Gamma = -\sum_{\gamma \in A} \gamma.$$

Then $A \cup \{\Gamma\}$ is a sequence of direction vectors that sums to 0 and spans $\mathbf{R}^3$. It follows by the Minkowski Existence Theorem there exists a convex polyhedron P such that the outward facing normal vectors of P are precisely parallel to the vectors in $A \cup \{\Gamma\}$. In particular, P contains faces $f_1, \ldots, f_n$ such that the normal at face f_i is γ_i, which is perpendicular to ζ_i as required.

□

Theorem 6 *There exists a convex polyhedron for which every steepest edge unfolding of contains a* 2*-local overlap.*

Proof. Let ϕ be the value from the statement of Lemma 4. By Lemma 5 there is a polyhedron P and set $\{\zeta_1, \ldots, \zeta_n\}$ of direction vectors such that every ζ is within angle ϕ of some ζ_i and for each ζ_i there is a face f_i of P such that ζ_i lies in the plane of f_i.

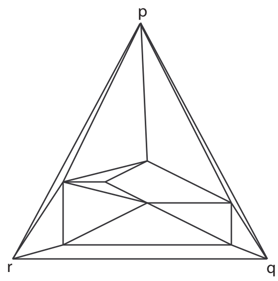

Figure 8: Embedding M_1 into a face (p, q, r) of a polyhedron.

We shall create a new convex polyhedron P_1 by embedding a copy of M_1 into face f_1. We perform this operation in the following steps.

1. Scale a copy of M_1 so that it fits in the interior of face f_1. Say this scaling is by a factor of λ_1. We can assume without loss of generality that $\lambda_1 < 1$.

2. Rotate M_1 in the plane of f_1 so that $\frac{e-f}{|e-f|} = \zeta_1$ (recall that e and f are vertices in M_1). This is always possible since ζ_1 lies in the plane of f_1.

3. Triangulate the space between the boundary of f_1 and the boundary of M_1. See Figure 8 for an illustration of such an embedding.

4. Raise all vertices of M_1 by a height of $\beta > 0$ above the plane of f_1, with β chosen small enough that the resulting polyhedron remains convex.

5. Raise vertices a and b by an additional height of $\lambda_1 \alpha_1$ above the plane of f_1, where α_1 is chosen small enough that the resulting polyhedron remains convex. We also choose α_1 small enough to satisfy Lemma 4. This transforms the embedded copy of M_1 into a copy of $M_1(\alpha_1)$, scaled by a factor of λ_1 (recall that our copy of M_1 was already scaled by λ_1).

Call the resulting polyhedron P_1. Then P_1 is convex and contains a scaled copy of $M_1(\alpha_1)$ where $\frac{e-f}{|e-f|} = \zeta_1$.

Repeat the above steps for faces $f_2, \ldots, f_n$, embedding a scaled copy of $M_1(\alpha_i)$ into each face. Call the resulting convex polyhedron P_n. We claim that P_n is the desired polyhedron. Well, for any direction vector ζ, there is a ζ_i such that the angle between ζ and ζ_i is less than ϕ. Consider the copy of $M_1(\alpha_i)$ embedded in face f_i. For this embedded terrain, the angle between ζ and $\frac{e-f}{|e-f|}$ is less than ϕ and α_i satisfies Lemma 4. Therefore, by Lemma 4, P_n will contain a 2-local overlap when cut along the steepest edge cut tree with direction ζ. We conclude that every steepest edge unfolding of P_n will contain an overlap. $\square$

6 Normal Order Unfoldings

We now consider a broad class of unfoldings: the *normal order* unfoldings. This class was motivated as a generalization of steepest edge unfoldings. We shall construct a polyhedron for which every normal order unfolding contains an overlap.

6.1 Definition

Let P be a convex polyhedron and choose a unit direction vector ζ. Given $f \in F(P)$, let n_f be the outward-facing unit normal for f. Let $z(f) = n_f \cdot \zeta$. Given $f, g \in F(P)$, we say g is *lower* than f if $z(g) \leq z(f)$. Informally, g is lower than f if n_g does not point more toward ζ than n_f does. Let $L(f)$ be the set of faces that are both adjacent to and lower than f.

Now consider a cut tree C of P, with corresponding adjacency tree A and unfolding P'. We say that the unfolding P' is a *normal order* unfolding if, for every $f \in F(P)$ with $L(f) \neq \emptyset$, there exists $g \in L(f)$ that is adjacent to f in P'. In other words, every face that is a adjacent to at least one lower face in the polyhedron P must be adjacent to a lower face in the unfolding P'.

We first show that this is a reasonable class of unfoldings by demonstrating that a normal order unfolding exists for any choice of P and ζ.

Proposition 7 *A convex polyhedron P has a normal order unfolding for any choice of direction vector ζ.*

Proof. We shall build an adjacency tree A that produces a normal order unfolding for P and ζ. For each f with $L(f) \neq \emptyset$, choose some $g_f \in L(f)$. Let A_1 be the graph with vertices $F(P)$ and edges $\{(f, g_f) : L(f) \neq \emptyset\}$. Then A_1 satisfies the property that every face f with $L(f) \neq \emptyset$ is adjacent to a lower face in A_1, but A_1 is not necessarily a spanning tree.

Suppose A_1 contains a cycle $f_1, f_2, \ldots, f_k, f_1$. Then each face in the cycle is lower than one of its adjacent faces, so we conclude $z(f_i) = z(f_j)$ for all $1 \leq i, j \leq k$. We can thus remove edge (f_1, f_2) from the graph and both f_1 and f_2 will still be adjacent to lower faces (via (f_k, f_1) and (f_2, f_3)). This process can be repeated to remove all cycles in A_1; call the resulting forest A_2. We then extend A_2 to a spanning tree A of $F(P)$ by adding any necessary edges. Then A retains the property that every face adjacent to at least one lower face in P is adjacent to a lower face in A, so A is an adjacency tree that produces a normal order unfolding. $\square$

6.2 Motivation

Our definition of normal order unfoldings is motivated by the steepest edge unfoldings. In Schlickenrieder's paper, there are examples of complex convex polyhedra with simple steepest edge unfoldings [6]. As an informal intuition, the success of these unfoldings appears to derive from their tendency to "expand outward" from a central point. It seems natural that a given convex polyhedron could be unfolded by expanding monotonically outward, and that such unfoldings would have a high probability of being simple. The definition of normal order unfoldings attempts to capture this notion of monotonicity.

A motivating question for the research presented in this paper was whether a simple normal order unfolding exists for every convex polyhedron. Unfortunately, despite our intuition, we shall now prove that this is not the case.

6.3 Construction

We shall construct a polyhedron for which every normal order unfolding contains an overlap. The method of construction is very similar to that for steepest edge unfoldings in Section 5. We simply modify the terrain M_1 to generate overlaps for all normal order unfoldings.

Consider the planar graph M_2 illustrated in Figure 9(a). An important thing to notice about this graph is that certain angles, illustrated in the figure, are all less than $\frac{\pi}{2}$.

Consider raising the interior vertices of M_2 by at most α in such a way that each face of M_2 remains planar. Call the resulting convex terrain $M_2(\alpha)$. See Table 1 for a particular instance of $M_2(\alpha)$.

Suppose that vector $(0, 0, 1)$ points toward the top of the page in Figure 9(a). Then all normal order unfoldings of $M_2(\alpha)$ with direction vector within some range of $(0, 0, 1)$ contain overlaps, as proved by the following lemma.

Lemma 8 *There exists a value of $\phi > 0$ such that if $\alpha > 0$ is sufficiently small and ζ is within an angle of ϕ from $(0, 0, 1)$ then any normal order unfolding of $M_2(\alpha)$ with respect to direction vector ζ contains an overlap.*

Proof. First suppose that $\zeta = (0, 0, 1)$. See Table 1 for a particular instance of $M_2(\alpha)$ with $\alpha = 0.03$, and the value of $z(f)$ for each face of $M_2(\alpha)$.

Consider a normal order unfolding of $M_2(\alpha)$. Note that what constitutes a valid normal order unfolding depends on the order of the faces according to face heights. Specifically, in Figure 9(a), the bold edges will not be cut in a normal order unfolding. These are the situations in which a face F is incident to only one other face G with $z(G) \leq z(F)$, and thus must be adjacent to that face in any normal order unfolding.

Let F denote the face (c, d, f, h). Note that there is a choice regarding which edges incident with F to cut. In a Normal Order unfolding, one of edges (d, f) or (c, d) must not be cut, since F must be adjacent to one of its adjacent lower faces; either (d, e, f) or (a, b, c, d, e).

Case 1: edge (c, d) is cut. Then a portion of the unfolding is as illustrated in Figure 9(b). Recall that angle $\angle icd$ is less than $\frac{\pi}{2}$ in M_2. Thus, if α is sufficiently small, the sum of the angles of faces (i, c, b) and (a, b, c, d, e) at vertex c will be greater than $\frac{3\pi}{2}$. Also, vertex d has degree 1 in the cut tree.

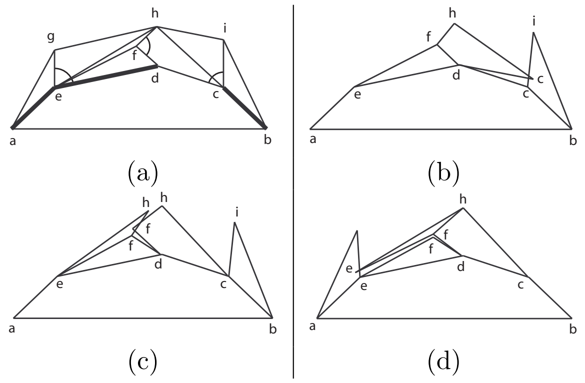

Figure 9: (a) The planar graph M_2. The marked edges are *not* cut in a normal order unfolding, and the marked angles are less than $\frac{\pi}{2}$. (b,c,d) Portions of the normal order unfoldings of $M_2(\alpha)$

But then, by Lemma 2, a 2-local overlap will occur in this unfolding of $M_2(\alpha)$ for sufficiently small α.

Case 2: edge (d, f) is cut. We now have two subcases.

Case 2.1: edge (f, h) is cut. Then the angles at f in faces (d, e, f) and (e, f, h) sum to more than $\frac{3\pi}{2}$ when α is sufficiently small. Also, vertex d has degree 1 in the cut tree. So, by Lemma 2, a 2-local overlap will occur in this unfolding when α is sufficiently small. See Figure 9(c).

Case 2.2: edge (f, h) is not cut. Then edge (f, e) must be cut. But then, taking curvatures sufficiently small, there will be an overlap between faces (a, e, g) and (e, f, h). See Figure 9(d).

This situation requires particular attention, since the occurrence of an overlap does not follow immediately from Lemma 2. In particular, the overlap is a 4-local overlap. However, the situation is quite similar to the conditions of Lemma 2, and the same form of argument can be applied to show that a 4-local overlap will occur for sufficiently small α. One can also consider a particular instance of M_2 and demonstrate numerically that an overlap occurs for sufficiently small α [4].

We conclude that there is no way to unfold terrain

Table 1: A particular instance of $M_2(\alpha)$, with vertex coordinates and face heights given $\zeta = (0, 0, 1)$.

vertex	coordinates	face F	$z(F)$
a	$(0, 0, 0)$	(a, b, c, d, e)	-0.010
b	$(10, 0, 0)$	(d, e, f)	-0.003
c	$(9, 0.02, 2)$	(c, d, f, h)	0.020
d	$(6, 0.03, 3)$	(b, c, i)	0.022
e	$(1, 0.02, 2)$	(c, h, i)	0.030
f	$(4, 0.03, 4)$	(e, f, h)	0.099
g	$(0.99, 0, 2.1)$	(a, e, g)	0.154
h	$(5, 0, 5)$	(e, g, h)	0.182
i	$(9.1, 0, 2.6)$		

$M_2(\alpha)$ while respecting the normal order induced by $\zeta = (0, 0, 1)$. Note that the ordering of heights of faces in $M_2(\alpha)$ will remain the same given minute perturbations of the terrain. Fix some sufficiently small α_0; then there exists some $\phi > 0$ such that the relative order of face heights of $M_2(\alpha_0)$ remains the same when ζ is perturbed by an angle of at most ϕ. In addition, ϕ can be chosen small enough that the ordering with this perturbed ζ remains the same for $M_2(\alpha)$ for all $0 < \alpha < \alpha_0$. But then the normal order unfoldings of $M_2(\alpha)$ are precisely those described above, which all contain overlaps.

This implies that if ζ is within ϕ of $(0, 0, 1)$, then any normal order unfolding of $M_2(\alpha)$ with direction ζ will contain an overlap. Further, our choice of α is independent of our choice of ϕ as long as α is sufficiently small. □

Theorem 9 *There exists a convex polyhedron P such that every normal order unfolding of P contains an overlap.*

Proof. This proof is very similar to that of Theorem 6, so the argument will be made briefly.

Choose α and ϕ to satisfy the conditions of Lemma 8. Let P be the polyhedron from Lemma 5 given angle ϕ. Embed a copy of M_2 into each of the faces $f_1, \ldots, f_n$ of P in the statement of Lemma 5, such that the top of M_2 (as in Figure 9(a)) faces toward ζ_i for each i. Raise the vertices of each copy of M_2 by a sufficiently small amount that the resulting polyhedron remains convex and the corresponding convex terrains are of the form $M_2(\alpha')$ where $\alpha' < \alpha$. Call the resulting convex polyhedron P'. Then for any direction vector ζ there is a corresponding copy of $M_2(\alpha')$ for which Lemma 8 applies, and hence any normal order unfolding with direction ζ will contain an overlap. Thus all normal order unfoldings of P' contain overlaps, as required. □

7 Conclusion

We have developed a methodology for constructing convex polyhedra for which a given class of unfoldings contains no simple unfoldings. This was used to negatively resolve conjectures by Fukuda and Schlickenrieder. We also applied this method to show that not every convex polyhedron has a simple normal order unfolding. This last counterexample serves to break the intuition that one can always construct a simple unfolding that "expands outwards" monotonically from a point.

This work leaves open a number of questions for future research. First, one might consider a class of unfoldings other than normal order unfoldings as a candidate for positively resolving Shephard's Conjecture. It is possible that a slightly different class of unfoldings could preserve the informal notion of creating starlike unfoldings, yet not fall to the type of counterexample presented in this paper.

Also, Lemma 2 gives only a set of sufficient conditions for a limited type of overlap. We could strengthen our ability to construct counterexamples by considering the more general notion of k-local overlaps. One could imagine generalizing Lemma 2 to a full characterization of necessary and sufficient conditions leading to k-local overlaps. Such a characterization would be a powerful tool and significant progress toward resolving Shephard's conjecture. However, even extending Lemma 2 to 3-local overlaps could prove enlightening in the study of convex polyhedra unfoldings.

Another question to pursue is whether or not it is true that every convex polyhedron has an unfolding that avoids 2-local overlaps. If no, the conclusion would be that Shephard's conjecture is false. If yes, then one might attempt to extend the result to k-local overlaps, which would positively resolve Shephard's conjecture. A possible first step in this line of research would be to determine a full characterization of the necessary and sufficient conditions in which a 2-local overlap occurs, expanding upon the results of Lemma 2.

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