Local Overlaps in Unfoldings of Polyhedra
Abstract

Anna Lubiw*   Brendan Lucier†

Abstract

We define a notion of local overlaps in polyhedron unfoldings. We use this concept to construct new examples of polyhedra that cannot be edge unfolded without overlap. In particular, we present a polyhedron with 16 triangular faces for which every unfolding contains a vertex with total face angle greater than 2π. We also construct an ununfoldable polyhedron with 9 convex faces, improving upon the previously best known bound of 13.

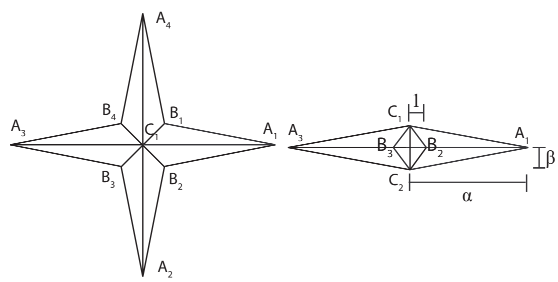
1 Introduction. An *edge unfolding* of a polyhedron is obtained by cutting some edges and unfolding the resulting surface into a connected planar piece. The edges that are cut in this process will form a spanning tree of the vertices called a *cut tree*. A *simple* edge unfolding is one that lies in the plane without overlap. If a polyhedron has no simple edge unfolding, we say that it is *ununfoldable*. A polyhedron is *starlike* if its entire surface is visible from a point in its interior. We are interested in ununfoldable polyhedra with convex faces.

In [1] it was shown that there exist ununfoldable polyhedra with convex faces. An example of an ununfoldable polyhedron with 24 convex faces was constructed, as was one with with 36 triangular faces. Later, Grünbaum presented an ununfoldable starlike polyhedron with only 13 convex faces and conjectured that 13 was minimal [2].

We shall approach the problem of finding small ununfoldable polyhedra by considering a particular type of overlap. A 1-*local overlap* is one in which the two faces are both coincident with a vertex in the unfolding. If a polyhedron has only convex faces, a 1-local overlap corresponds to a vertex in an unfolding having a total face angle greater than 2π.

We shall construct a starlike polyhedron with 16 triangular faces for which every edge unfolding contains a 1-local overlap. We then modify this example to form an ununfoldable starlike polyhedron with 9 convex faces.

2 1-Local Overlaps are Unavoidable. We now present our example of an ununfoldable polyhedron and sketch the ideas behind the proof that it cannot avoid 1-local overlaps.



(a) Top View   (b) Side View

Figure 1: Polyhedron $P_4(\alpha, \beta)$

Our polyhedron is the four-pointed star shown in Figure 1, which depends on parameters α and β. We think of α as being large and β as being small. Then our polyhedron consists of four spikes, having endpoints $A_1 = (\alpha, 0, 0)$, $A_2 = (0, -\alpha, 0)$, $A_3 = (-\alpha, 0, 0)$, and $A_4 = (0, 0, \alpha)$. These spikes intersect pairwise at vertices $B_1 = (1, 1, 0)$, $B_2 = (1, -1, 0)$, $B_3 = (-1, -1, 0)$, and $B_4 = (-1, 1, 0)$.

*School of Comput. Sci., University of Waterloo, Ontario, Canada. alubiw@uwaterloo.ca.

†School of Comput. Sci., University of Waterloo, Ontario, Canada. Supported by NSERC. blucier@uwaterloo.ca.

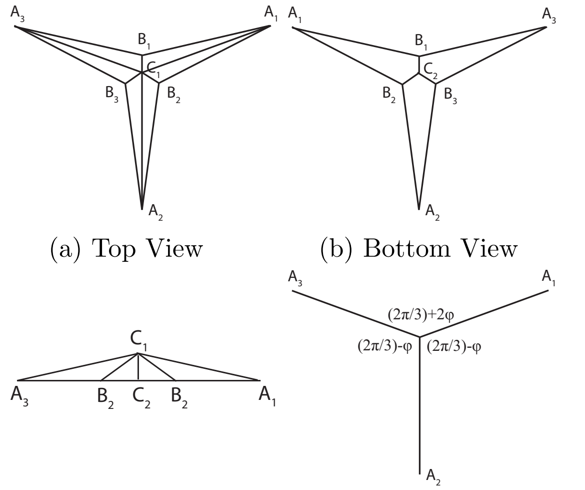
Finally, there are central points $C_1 = (0, 0, \beta)$ and $C_2 = (0, 0, -\beta)$ that are connected to all other vertices, but not to each other. We shall denote this polyhedron by $P_4(\alpha, \beta)$.

For sufficiently small β and sufficiently large α, each face angle at each B_i will be larger than $\frac{2\pi}{3}$. Setting $\alpha = 10$ and $\beta = 1$ is sufficient for this purpose, so we now consider the polyhedron $P_4(10, 1)$.

A 1-local overlap occurs precisely when a vertex in an unfolding has a total face angle greater than 2π, and hence its incident faces cannot lie in the plane without overlap. To see that all edge-unfoldings of $P_4(10, 1)$ contain 1-local overlaps, consider the faces incident with vertices B_i. Since the face angles at each B_i are greater than $\frac{2\pi}{3}$, no three faces incident with a given B_i can remain adjacent in an unfolding without creating an angle greater than 2π. This forces two opposing edges incident with each B_i to be cut. In addition, there must be two cuts incident with each of C_1 and C_2 to avoid 1-local overlaps, since these vertices have negative curvature. However, no cut tree can satisfy all of these cut requirements: all possibilites lead to cycles in cut edges. Thus any edge unfolding of $P_4(10, 1)$ must contain a 1-local overlap.

3 A Small Ununfoldable Polyhedron. We now modify P_4 to create a polyhedron with 9 faces for which every edge unfolding contains an overlap. This polyhedron is the three-pointed star shown in Figure 2. The polyhedron is not 3-way symmetric: the angles between the spikes are made unequal according to a parameter ϕ. See Figure 2(d). We also flatten one side of the polyhedron, combining 6 triangular faces into 3 quadrilaterals as illustrated in Figure 2(b). We call this polyhedron $P_3^{\phi}(\alpha, \beta)$, where α is the distance of each vertex A_i from the origin and β is the distance of C_1 from the origin.

The motivation for this construction is that the face angles at B_2 and B_3 will be larger than $\frac{2\pi}{3}$ for sufficiently extreme values of α and β. Then, just as with P_4, two opposing edges for B_2 and B_3 must be cut and there must be two cuts incident with C_1. These requirements eliminate all but a few possible unfoldings. The remaining possibilities can be shown to contain overlaps by a straightforward case analysis. The details of the argument are given in [3]. We conclude that $P_3^{\phi}(\alpha, \beta)$ is ununfoldable for appropriate choices of parameters. In particular, this is true when $\phi = \frac{\pi}{18}$, β is sufficiently small, and α is sufficiently large.



(a) Top View (b) Bottom View (c) Side View (d) Angle ϕ

Figure 2: The polyhedron $P_3^{\phi}(\alpha, \beta)$

The polyhedron $P_3^{\phi}(\alpha, \beta)$ has the unfortunate property that its three quadrilateral faces are coplanar. However, this minor degeneracy can be removed by shifting vertices A_1, A_2, A_3, and C_2 by a sufficiently small amount. See [3] for further detail. The resulting polyhedron is completely nondegenerate, starlike, and ununfoldable.

References

[1] M. Bern, E. D. Demaine, D. Epstein and E. Kuo, A. Mantler and J. Snoeyink. Ununfoldable polyhedra with convex faces. *Computational Geometry: Theory and Applications*, 24(2):51–62,2003.

[2] B. Grünbaum. No-net polyhedra. *Geombinatorics*, 11:111–114, 2002.

[3] B. Lucier. Unfolding and Reconstructing Polyhedra. MMath Thesis in preparation, University of Waterloo, 2005.