Available online at www.sciencedirect.com
ScienceDirect
Journal of Symbolic Computation 43 (2008) 92–117
Journal of Symbolic Computation
www.elsevier.com/locate/jsc

Computing singular points of plane rational curves

Falai Chen[a], Wenping Wang[b], Yang Liu[b]

[a] *Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, PR China*
[b] *Department of Computer Science, The University of Hong Kong, Pokfulam Road, Hong Kong, China*

Received 11 August 2006; accepted 3 October 2007
Available online 24 October 2007

Abstract

We compute the singular points of a plane rational curve, parametrically given, using the *implicitization* matrix derived from the μ-basis of the curve. It is shown that *singularity factors*, which are defined and uniquely determined by the elementary divisors of the implicitization matrix, contain all the information about the singular points, such as the parameter values of the singular points and their multiplicities. Based on this observation, an efficient and numerically stable algorithm for computing the singular points is devised, and inversion formulae for the singular points are derived. In particular, high order singular points can be detected and computed effectively. This approach based on singularity factors can also determine whether a rational curve has any non-ordinary singular points that contain singular points in its infinitely near neighborhood. Furthermore, a method is proposed to determine whether a singular point is ordinary or not. Finally, a conjecture in [Chionh, E.-W., Sederberg, T.W., 2001. On the minors of the implicitization bézout matrix for a rational plane curve. Computer Aided Geometric Design 18, 21–36] regarding the multiplicity of the singular points of a plane rational curve is proved.
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Keywords: Rational parametric curve; Singular point; Inversion formula; Implicitization; μ-basis

1. Introduction

We are interested in finding the singularities of real planar rational curves parametrically given (although the results can be carried out in general for rational curves over algebraically closed field of characteristic zero). A *singular point* of a curve is a point on the curve where the tangent line of the curve is not uniquely defined. The singularities of a curve represent shape features known as cusps or self-intersections. Thus detection of singularities helps to determine the

E-mail addresses: chenfl@ustc.edu.cn (F. Chen), wenping@cs.hku.hk (W. Wang), yliu@cs.hku.hk (Y. Liu).

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doi:10.1016/j.jsc.2007.10.003

geometric shape and topology of real curves, which has wide-ranging applications in computer aided geometric design. Furthermore, singularities of curves are significant in theory. A recent discovery about curve singularities was made by Sederberg and one of the present authors (Chen and Sederberg, 2002); they show that singularities of a curve are closely related with the μ-basis of the curve and thus affect the result of implicitization, a procedure for converting a parametric form of a rational curve to an implicit form.

Consider an algebraic curve $F(x, y, w) = 0$ in its homogeneous form. A point $P_0 = (x_0, y_0, w_0)$ is a singular point if and only if

$$F_x(x_0, y_0, w_0) = 0, \qquad F_y(x_0, y_0, w_0) = 0, \qquad F_w(x_0, y_0, w_0) = 0.$$

P_0 is called an *r-fold singular point* or a singular point with *multiplicity* r of $F = 0$ if all derivatives of F of order $r - 1$ are zero at P_0 and at least one rth derivative of F does not vanish at P_0.

A singular point P_0 is said to be *ordinary* if all the tangents of $F = 0$ at P_0 are distinct; otherwise it is called a *non-ordinary* singular point. For analyzing non-ordinary singular points, the notion of neighboring points can be introduced (see Chapter III in Walker (1950)). If a singular point P_0 has any singular point in its infinitely near neighborhood, then P_0 is necessarily a non-ordinary singular point, but the converse is not true; for example, a cusp on a rational plane cubic curve does not have any singular point in its infinitely near neighborhood but is non-ordinary, since the two tangents of the curve at the cusp are identical.

A rational curve $P(t)$ can be represented implicitly by $F(x, y, w) = 0$, for a unique homogeneous polynomial F up to a multiplicative constant. However, we analyze and compute the singularities of a rational curve directly from its parametric form $P(t)$, without resorting to its implicit form $F(x, y, w) = 0$.

Previous works on detecting the singular points on a cubic curve have been done by Wang (1981), Su and Liu (1983), Stone and Derose (1989) and Sakai (1999). They derived necessary and sufficient conditions for the existence of cusps and self-intersections on a plane cubic Bézier curve. For general degree rational curves, methods exist to detect the cusps and inflection points (Manocha and Canny, 1992; Li and Cripps, 1997). However, their methods are not applicable to computing all the singular points of a general rational curve.

There are a few methods in the literature on computing the singular points of a rational plane curve. One standard way is to convert the parametric equation of the curve into an implicit equation $F(x, y, w) = 0$ (in homogeneous form), and then find the singular points by solving the system of equations (Walker, 1950):

$$F_x(x, y, w) = 0, \qquad F_y(x, y, w) = 0, \qquad F_w(x, y, w) = 0.$$

Other methods for finding the singular points include Peterson's method (Peterson, 1917), where a nonlinear system of equations with two variables need to be solved. Abhyankar's method (Abhyankar, 1990, p. 153) involves Taylor t-resultant and applies only to polynomial curves, and it was later extended to rational curves by Gutierrez et al. (2002). Recently, a more direct approach was given by Chionh and Sederberg (2001). They compute the parameter values of the singular points by intersecting the parametric curve with the first minor of the implicitization Bézout matrix. However, this approach suffers from serious numerical problems, since it is not trivial to accurately and robustly find the roots (often with high multiplicities) of a high degree polynomial. Furthermore, this method does not compute explicitly the multiplicity of the singular points.

In this paper, we improve the method by Chionh and Sederberg by giving an efficient and numerically stable algorithm for computing the singular points of a plane rational parametric curve. We start from the implicitization matrix derived from the μ-basis of the rational curve, and its *singularity factors*, which are defined in terms of the elementary divisors of the implicitization matrix as a polynomial matrix (see the definition in Section 3). We show that the singularity factors contain all information about the singular points, such as the parameter values of the singular points and their multiplicities. Based on this result, a numerically stable algorithm is presented to compute the singular points of the curve, and *inversion formulae* (see the definition in Section 3) for singular points are derived. Furthermore, a necessary and sufficient condition is presented for a rational curve containing only ordinary singular points, a method is provided to determine if a singular point is ordinary or not. A conjecture in Chionh and Sederberg (2001) regarding the multiplicity of singular points is also proved. On finishing this paper, we just found a paper by S. Pérez-Díaz which solves the similar problem but with a totally different technique (Pérez-Díaz, 2007).

The remainder of the paper is organized as follows. Section 2 provides some preliminaries about the μ-basis and implicitization matrix of a rational plane curve. Section 3 studies the relationship between the singular points and the singularity factors of the implicitization matrix. Section 4 discusses details on the computation of singular points based on the result in Section 3. Section 5 summarizes the results for the singular points of low degree parametric curves. Finally, in Section 6, we conclude the paper with discussions about further research.

2. Preliminaries

Let

$$\mathbf{P}(t) = (a(t), b(t), c(t)) \tag{1}$$

be a rational plane curve in projective coordinates, where $a(t), b(t), c(t)$ are relatively prime polynomials, the maximum degree of which is n. We assume that the parametrization is proper, that is, with the exception of a finite number of points, every point of $P(t)$ corresponds to a unique parameter value of t.

A *moving line* $\mathbf{L}(t)$ is a family of lines with one free parameter t (Sederberg and Chen, 1995):

$$\mathbf{L}(t) := A(t)x + B(t)y + C(t)w = 0, \tag{2}$$

where $A(t)$, $B(t)$, and $C(t)$ are polynomials. We also denote $\mathbf{L}(t)$ in a vector form $(A(t), B(t), C(t))$. A moving line is said to *follow* the rational curve (1) if the following equation holds,

$$\mathbf{L}(t) \cdot \mathbf{P}(t) = A(t)a(t) + B(t)b(t) + C(t)c(t) \equiv 0. \tag{3}$$

A moving line $\mathbf{L}(t)$ is said to have an *axis* $\mathbf{P}_0$ if $\mathbf{L}(t) \cdot \mathbf{P}_0 = 0$ for any parameter t, that is, all the lines in the family $\mathbf{L}(t)$ pass through the point $\mathbf{P}_0$. If a moving line has an axis, then it is called an *axial* moving line. One important result about moving lines is the following.

Proposition 1 (*Cox et al., 1998*). *There exist two moving lines* $\mathbf{p}(t)$ *and* $\mathbf{q}(t)$ *of degree* μ *(*$\mu \leq \lfloor n/2 \rfloor$*) and* $n - \mu$ *respectively, such that*

$$\mathbf{P}(t) = \kappa \mathbf{p}(t) \times \mathbf{q}(t).$$

Here κ is some nonzero constant. Furthermore, $\mathbf{p}(t_0)$ *and* $\mathbf{q}(t_0)$ *are linearly independent for any parameter value* t_0.

The moving lines $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are called the *μ-basis* of the rational curve $\mathbf{P}(t)$; equivalently, we also call $p(x, y, w; t) = \mathbf{p}(t) \cdot \mathbf{X}$ and $q(x, y, w; t) = \mathbf{q}(t) \cdot \mathbf{X}$ the μ-basis, where $\mathbf{X} = (x, y, w)$. Efficient algorithms exist for computing the μ-basis, see Chen and Wang (2002), for example. Once the μ-basis is known, the implicit equation of a rational curve $\mathbf{P}(t)$ can be obtained by simply taking the resultant of p and q with respect to t.

Proposition 2 (*Cox et al., 1998*). *Let* $p(x, y, w; t)$ *and* $q(x, y, w; t)$ *be the μ-basis of the rational curve* $\mathbf{P}(t)$. *Then an implicit equation (in homogeneous form) of* $\mathbf{P}(t)$ *is given by*

$$F(x, y, w) := \mathbf{Res}(p, q; t) = 0.$$

Assume that

$$p = \sum_{i=0}^{\mu} \hat{p}_i(x, y, w)t^i, \qquad q = \sum_{i=0}^{n-\mu} \hat{q}_i(x, y, w)t^i,$$

where $\hat{p}_i$ and $\hat{q}_i$ are homogeneous linear functions in x, y, w. The implicitization matrix of p and q can be expressed as (Cox et al., 1998)

$$\hat{B}(x, y, w) = \begin{pmatrix} R_{1,1} & \cdots & R_{1,n-\mu} \\ \vdots & \vdots & \vdots \\ R_{n-\mu,1} & \cdots & R_{n-\mu,n-\mu} \end{pmatrix}, \tag{4}$$

where

$$R_{ij} = \begin{cases} \hat{p}_{n-\mu-i-j+1}, & n - 2\mu - i + 1 \le j \le n - \mu - i + 1 \\ 0, & \text{otherwise}, \end{cases}$$

for $i = 1, 2, \ldots, n - 2\mu$, and

$$R_{ij} = \sum_{\substack{k_1 \le n-\mu-i \\ 2n-3\mu-i+1 \le k_2 \le n-\mu \\ k_1+k_2=2(n-\mu)-i-j+1}} \hat{p}_{k_1}\hat{q}_{k_2} - \sum_{\substack{k_1 \le 2n-3\mu-i \\ n-\mu-i+1 \le k_2 \le \mu \\ k_1+k_2=2(n-\mu)-i-j+1}} \hat{p}_{k_2}\hat{q}_{k_1},$$

for $i = n - 2\mu + 1, \ldots, n - \mu$. Note that the first $n - 2\mu$ rows of $\hat{B}(x, y, w)$ is linear in x, y, w and the last μ rows are quadratic in x, y, w. By Proposition 2,

$$F(x, y, w) = \det(\hat{B}(x, y, w)). \tag{5}$$

The implicit equation of $\mathbf{P}(t)$ can also be written as the determinant of an $\mu \times \mu$ matrix.

Proposition 3 (*Cox et al., 1998*). *Let* C_p *be the companion matrix of polynomial* p. *Then the implicit equation of* $\mathbf{P}(t)$ *is given by*

$$F(x, y, w) = \hat{p}_\mu^{n-\mu} \det(q(C_p)) = 0.$$

Here

$$C_p = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\hat{p}_0/\hat{p}_\mu & \ldots & -\hat{p}_{\mu-2}/\hat{p}_\mu & -\hat{p}_{\mu-1}/\hat{p}_\mu \end{pmatrix}_{\mu\times\mu}.$$

The μ-basis has the property that under an invertible projective transformation of the rational curve $\mathbf{P}(t)$, the μ-basis is also changed by some invertible projective transformation.

Proposition 4. *Let $\delta = (t_{ij})_{3\times 3}$ be an invertible matrix and $\mathbf{P}'(t) = \mathbf{P}(t)\delta$ be the projective transformation curve of $\mathbf{P}(t)$. Then the μ-basis of $\mathbf{P}'(t)$ is given by*

$$\mathbf{p}'(t) = \mathbf{p}(t)\delta^{(-T)}, \qquad \mathbf{q}'(t) = \mathbf{q}(t)\delta^{(-T)}.$$

Proof. Since δ is invertible, $\deg(\mathbf{p}'(t)) = \deg(\mathbf{p}(t))$ and $\deg(\mathbf{q}'(t)) = \deg(\mathbf{q}(t))$. On the other hand, it is direct to verify that

$$\mathbf{p}'(t) \times \mathbf{q}'(t) = (\mathbf{p}(t)\delta^{(-T)}) \times (\mathbf{q}(t)\delta^{(-T)}) = (\mathbf{p}(t) \times \mathbf{q}(t))\delta/\det(\delta) = \mathbf{P}'(t)/\det(\delta).$$

Hence, $\mathbf{p}'(t)$ and $\mathbf{q}'(t)$ form a μ-basis of $\mathbf{P}'(t)$. □

The value μ in the μ-basis classifies all the degree n rational curves into $\lfloor n/2\rfloor+1$ classes. For $\mu = 0$, the rational curve is a multiply traced line. $\mu = 1$ if and only if the rational curve contains a singular point of order $n-1$. However, when $\mu \geq 2$, very little is known about the singular points of the rational curve. In this paper, we will study relationships between the singular points and the value μ, and provide an algorithm, based on the implicitization matrix derived from the μ-basis, to compute all the singular points of a rational curve.

3. Singular points of a rational curve

In this section, we study the singular points of a planar rational parametric curve by the μ-basis. To allow the case $t = \infty$, we use homogeneous parameters $t : u$ instead of t. Thus the curve $\mathbf{P}(t)$ becomes

$$\mathbf{P}(t,u) = (a(t,u), b(t,u), c(t,u)),$$

where $a(t,u)$, $b(t,u)$ and $c(t,u)$ are homogenized from $a(t)$, $b(t)$ and $c(t)$ by $a(t/u)u^n$, $b(t/u)u^n$ and $c(t/u)u^n$ respectively. Similarly, the μ-basis $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are homogenized to $\mathbf{p}(t,u)$ and $\mathbf{q}(t,u)$.

Assume that

$$\mathbf{p} := \mathbf{p}(t,u) = (p_1(t,u), p_2(t,u), p_3(t,u)),$$
$$\mathbf{q} := \mathbf{q}(t,u) = (q_1(t,u), q_2(t,u), q_3(t,u))$$

are the μ-basis of the rational curve $\mathbf{P}(t,u)$ whose degrees are μ and $n-\mu$ respectively, where $p_i(t,u)$ and $q_i(t,u)$, $i = 1, 2, 3$, are degree n homogeneous polynomials. Let

$$p := p(x,y,w;t,u) = \mathbf{p}(t,u)\cdot(x,y,w) = p_1x + p_2y + p_3w,$$
$$q := q(x,y,w;t,u) = \mathbf{q}(t,u)\cdot(x,y,w) = q_1x + q_2y + q_3w$$

with $\mu \leq \lfloor n/2\rfloor$. We will show that the only possible singular point of order higher than μ of $\mathbf{P}(t,u)$ has order $n-\mu$.

Theorem 1. *If $\mathbf{P}(t,u)$ has a singular point $\mathbf{P}_0$ of order $r > \mu$, then $r = n-\mu$ and $\mathbf{p}$ is an axial moving line with $\mathbf{P}_0$ being its axis. Furthermore, the curve $\mathbf{P}(t,u)$ has at most one singular point $\mathbf{P}_0$ of order $r > \mu$.*

Proof. Without loss of generality, we may assume that the singular point $\mathbf{P}_0$ is at the origin. Then there exist homogeneous polynomials f, g, h such that

$$a(t,u) = f(t,u)g(t,u), \qquad b(t,u) = f(t,u)h(t,u),$$

where $\deg(f) = r$, $\deg(g) = \deg(h) = n - r$, $GCD(g,h) = 1$ and $GCD(f,c) = 1$.

Since $\mathbf{p}$ is a moving line, it follows that $\mathbf{p}(t,u) \cdot \mathbf{P}(t,u) = p_1 fg + p_2 fh + p_3 c \equiv 0$. From $GCD(f,c) = 1$, we have $f | p_3$. Since $\deg(f) = r > \mu = \deg(p_3)$, it follows that $p_3 = 0$ and $p = p_1 x + p_2 y$, which is an axial moving line with the origin being the axis. Furthermore, from $p_1 g + p_2 h = 0$ and $GCD(g,h) = 1$, there exists a homogeneous polynomial $k(t,u)$ such that $p_1 = kh$ and $p_2 = -kg$. Thus $p = k(hx - gy)$. By Proposition 1, p is the lowest degree moving line, therefore k must be a nonzero constant. So we have $r = n - \mu$.

Since p cannot have more than one axis, $\mathbf{P}(t,u)$ cannot have more than one singular point of order $r > \mu$. This completes the proof. □

By Theorem 1, we immediately have

Corollary 1. *$\mu = 1$ if and only if the rational curve $\mathbf{P}(t,u)$ has a singular point of order $n - 1$.*

Proof. Suppose that $\mathbf{P}(t,u)$ has a singular point of order $n-1$. Then by Theorem 1, $n-1 = n-\mu$. It follows that $\mu = 1$. Conversely, suppose that $\mu = 1$. Let P_0 be an order r singular point of $\mathbf{P}(t,u)$. Since $r > \mu = 1$, by Theorem 1, $r = n - \mu = n - 1$, and such a P_0 is unique. □

Theorem 2. *The rational curve $\mathbf{P}(t,u)$ has an order $n - \mu$ singular point (with $\mu < n - \mu$) if and only if p is an axial moving line.*

Proof. The proof of Theorem 1 implies that if $\mathbf{P}(t,u)$ has a singular point of order $n - \mu$, then p is an axial moving line. Conversely, suppose that p is an axial moving line with $\mathbf{P}_0$ being the axis. Then $\mathbf{p}(t,u) \cdot \mathbf{P}_0 \equiv 0$, and thus for any $t : u$ satisfying $\mathbf{q}(t,u) \cdot \mathbf{P}_0 = 0$, one must have $\mathbf{P}_0 = k\mathbf{p}(t,u) \times \mathbf{q}(t,u) = k\mathbf{P}(t,u)$ for some nonzero constant k, that is, $\mathbf{P}_0$ is a point on curve $\mathbf{P}(t,u)$. Since there are $n - \mu$ values of $t : u$ satisfying $\mathbf{q}(t,u) \cdot \mathbf{P}_0 = 0$ over the field of complex numbers, $\mathbf{P}_0$ is a singular point of order $n - \mu$. □

The above theorem provides a simple way to detect and compute a high order singular point of a rational curve. We write $p = \sum_{i=0}^{\mu} \hat{p}_i(x,y,w) t^i u^{\mu-i}$, and if all the lines $\hat{p}_i(x,y,w) = 0$, $i = 0, 1, \ldots, \mu$, intersect at the same point $\mathbf{P}_0$, then p is an axial moving line with $\mathbf{P}_0$ being the axis, and $\mathbf{P}_0$ is the singular point of order $n - \mu$.

Next we explore methods to compute all the singular points of a rational curve. Before going on, we need to state the following well-known result (Fuhrmann, 1996).

Lemma 1. *Let $f(t,u)$ and $g(t,u)$ be two homogeneous polynomials with degree m and n ($n \geq m$) respectively. Let $Bez(f,g)$ denote the Bézout resultant matrix of f and g. Then f and g have a greatest common divisor of degree r if and only if* $\operatorname{rank}(Bez(f,g)) = n - r$.

Suppose that $\mathbf{P}_0 = (x_0, y_0, w_0)$ is a singular point of order r on $\mathbf{P}(t,u)$. An *inversion formula* of $\mathbf{P}_0$ is a homogeneous polynomial whose roots (including multiplicities) are the parameter values corresponding to $\mathbf{P}_0$, i.e., the fibre of the singular point $\mathbf{P}_0$ via the rational map induced by $\mathbf{P}(t,u)$. Since $\mathbf{p}(t,u)$ and $\mathbf{q}(t,u)$ follow the rational curve $\mathbf{P}(t,u)$, the two polynomials $f(t,u) := \mathbf{P}_0 \cdot \mathbf{p}(t,u)$ and $g(t,u) := \mathbf{P}_0 \cdot \mathbf{q}(t,u)$ must have the inversion formula of $\mathbf{P}_0$ as a common factor. In fact, we have

Lemma 2. *Let* $\mathbf{P}_0 = (x_0, y_0, w_0)$ *be an order* r *singular point of* $\mathbf{P}(t, u)$ *and* $h(t, u)$ *be the inversion formula of* $\mathbf{P}_0$*. Then the GCD of* $f(t, u)$ *and* $g(t, u)$ *is* $h(t, u)$*.*

Proof. Without loss of generality, we assume that $\mathbf{P}_0 = (0, 0, 1)$. Then $f(t, u) = p_3$, $g(t, u) = q_3$ and there exist polynomials $\bar{f}(t, u)$ and $\bar{g}(t, u)$ such that

$$a(t, u) = \bar{f}(t, u)h(t, u), \qquad b(t, u) = \bar{g}(t, u)h(t, u),$$

where $GCD(\bar{f}, \bar{g}) = 1$ and $GCD(h, c) = 1$. From

$$\mathbf{P}(t, u) \cdot \mathbf{p}(t, u) \equiv 0, \qquad \mathbf{P}(t, u) \cdot \mathbf{q}(t, u) \equiv 0,$$

we have

$$(p_1\bar{f} + p_2\bar{g})h + p_3c \equiv 0, \qquad (q_1\bar{f} + q_2\bar{g})h + q_3c \equiv 0. \tag{6}$$

Since $GCD(h, c) = 1$, we get $h|p_3$ and $h|q_3$. Thus, $h|GCD(p_3, q_3)$, that is, $h|GCD(f, g)$.

Let $GCD(f, g) = kh$. We claim that k is a nonzero constant. Assume the contrary. Let $t_0 : u_0$ be a zero of k. Then it follows from Eq. (6) that

$$p_1(t_0, u_0)\bar{f}(t_0, u_0) + p_2(t_0, u_0)\bar{g}(t_0, u_0) = 0$$
$$q_1(t_0, u_0)\bar{f}(t_0, u_0) + q_2(t_0, u_0)\bar{g}(t_0, u_0) = 0.$$

Since $\mathbf{p}$ and $\mathbf{q}$ are linearly independent for any parameter value and $p_3(t_0, u_0) = q_3(t_0, u_0) = 0$, $p_1(t_0, u_0)q_2(t_0, u_0) - p_2(t_0, u_0)q_1(t_0, u_0) \neq 0$. Hence, $\bar{f}(t_0, u_0) = \bar{g}(t_0, u_0) = 0$, which implies that $\bar{f}$ and $\bar{g}$ have a common factor, a contradiction. Thus k is a nonzero constant, and therefore $h = GCD(f, g)$. □

By Lemmas 1 and 2, we immediately have

Theorem 3. $\mathbf{P}_0 = (x_0, y_0, w_0)$ *is an order* r *singular point of the rational curve* $\mathbf{P}(t, u)$ *if and only if* $\operatorname{rank}(\hat{B}(x_0, y_0, w_0)) = n - \mu - r$*, where the matrix* $\hat{B}(x, y, w)$ *is defined in* (4)*.*

Therefore, to compute the singular points of a rational curve $\mathbf{P}(t, u)$, we need to explore the matrix

$$B(t, u) := \hat{B}(a(t, u), b(t, u), c(t, u)). \tag{7}$$

We still call $B(t, u)$ the *Bézout matrix* or *implicitization matrix* derived from the μ-basis. The matrix $B(t, u)$ provides all the information about the singular points.

Corollary 2. $\operatorname{rank}(B(t, u)) = n - \mu - 1$ *except at parameter values corresponding to the singular points of* $\mathbf{P}(t, u)$*.*

Proof. Since $\det(\hat{B}(x, y, w))$ gives the implicit equation of the rational curve $\mathbf{P}(t, u)$, $\det(B(t, u)) \equiv 0$, that is, $\operatorname{rank}(B(t, u)) \leq n - \mu - 1$. On the other hand, if $\operatorname{rank}(B(t_0, u_0)) < n - \mu - 1$ at some parameter value $t_0 : u_0$, then by Theorem 3, $t_0 : u_0$ corresponds to a singular point of $\mathbf{P}(t, u)$. □

The next corollary follows directly from Theorem 3.

Corollary 3. $\mathbf{P}(t_0, u_0)$ *is a singular point of order* r *if and only if* $\operatorname{rank}(B(t_0, u_0)) = n - \mu - r$.

Recall that the order k determinant factor of a matrix is the GCD of all the order k minors of the matrix. Let D_k be the determinant factor of $B(t, u)$, $k = 1, 2, \ldots, n - \mu$.

Lemma 3. *Let $h(t,u)$ be the inversion formula of an order r singular point $\mathbf{P}_0$ of $\mathbf{P}(t,u)$. Then $h(t,u)|D_{n-\mu-r+1}$.*

Proof. Without loss of generality, we assume that the singular point is at the origin, i.e., $\mathbf{P}_0 = (0,0,1)$. Then there exist polynomials $\bar{f}(t,u)$ and $\bar{g}(t,u)$ such that

$$a(t,u) = \bar{f}(t,u)h(t,u), \qquad b(t,u) = \bar{g}(t,u)h(t,u),$$

where $GCD(\bar{f},\bar{g}) = 1$, $GCD(h,c) = 1$. Let $e_{ij}(t,u)$ be the (i,j) element of matrix $B(t,u)$. When the corresponding element of $\hat{B}(x,y,w)$ is linear in x,y,w, $e_{ij}(t,u)$ takes the form

$$e_{ij}(t,u) = \alpha a(t,u) + \beta b(t,u) + \gamma c(t,u) = (\alpha\bar{f} + \beta\bar{g})h + \gamma c.$$

When the (i,j) element of $\hat{B}(x,y,w)$ is quadratic in x,y,w,

$$\begin{aligned} e_{ij}(t,u) &= \alpha a^2 + \beta ab + \gamma b^2 + \delta ac + \lambda bc + \nu c^2 \\ &= (\varrho\bar{f}^2 + \sigma\bar{f}\bar{g} + \varsigma\bar{g}^2)h^2 + (\delta\bar{f} + \lambda\bar{g})hc + \nu c^2. \end{aligned}$$

Thus, in general, $B(t,u)$ can be written as

$$B(t,u) = hG(t,u) + cH(t,u),$$

where $G(t,u)$ and $H(t,u)$ are polynomial matrices of order $n-\mu$.

$$\begin{aligned} &G(t,u) = (G_{20}\bar{f}^2 + G_{11}\bar{f}\bar{g} + G_{02}\bar{g}^2)h + (G_{10}\bar{f} + G_{01}\bar{g})c, \\ &H(t,u) = DH_{10}, \end{aligned}$$

with G_{ij} and H_{ij} being constant matrices of order $n-\mu$ and $D = \mathrm{diag}(1,\ldots,1,c,\ldots,c)$, where the first $n-2\mu$ elements are 1. Let $t_0 : u_0$ be a zero of $h(t,u)$. Since $c(t_0,u_0) \neq 0$, by Theorem 3,

$$\mathrm{rank}(H_{10}) = \mathrm{rank}(H(t_0,u_0)) = \mathrm{rank}(B(t_0,u_0)) = n-\mu-r.$$

Let $B_{n-\mu-r+1}(t,u)$ be an order $n-\mu-r+1$ submatrix of $B(t,u)$. Then

$$B_{n-\mu-r+1}(t,u) = hG_{n-\mu-r+1}(t,u) + cH_{n-\mu-r+1}(t,u),$$

where $G_{n-\mu-r+1}(t,u)$ and $H_{n-\mu-r+1}(t,u)$ are the corresponding order $n-\mu-r+1$ submatrices of G and H, respectively. Thus

$$\det(B_{n-\mu-r+1}) = \det(G_{n-\mu-r+1})h^{n-\mu-r+1} + \cdots + c^{n-\mu-r+1}\det(H_{n-\mu-r+1}).$$

Since $\mathrm{rank}(H_{10}) = n-\mu-r$, it is easy to see that the last term in the above equation is identically zero. Therefore, $h|\det(B_{n-\mu-r+1})$. Hence, $h|D_{n-\mu-r+1}$. □

Recall that a polynomial matrix is said to be *invertible* if it is a unit in the matrix ring, i.e. if its determinant does not vanish and its inverse is also a polynomial matrix. The following discussion involves the Smith form of a polynomial matrix, a concept that can be found in any standard text on Linear Algebra (e.g., Lancaster and Tismenetsky (1985)).

Lemma 4. *There exist unique polynomials $\bar{d}_i(t)$, $i = 2,3,\ldots,n-\mu$, such that*

$$R_1(t)B(t,1)S_1(t) = \mathrm{diag}(\bar{d}_{n-\mu}(t), \bar{d}_{n-\mu}(t)\bar{d}_{n-\mu-1}(t), \ldots, \bar{d}_{n-\mu}(t)\ldots\bar{d}_2(t), 0),$$

for some invertible matrices $R_1(t)$, $S_1(t)$. Furthermore, $\mathbf{P}(t_0,1)$ is a singular point of order r if and only if $\bar{d}_r(t_0) = 0$ and $\bar{d}_i(t_0) \neq 0$, $i = r+1,\ldots,n-\mu$.

Proof. The first part follows from the Smith form of $B(t, 1)$, which is a diagonal form each entry of which is a product of elementary divisors of $B(t, 1)$ (Lancaster and Tismenetsky, 1985).

The second part follows from the fact that

$$\text{rank}(B(t, 1)) = \text{rank}(\text{diag}(\bar{d}_{n-\mu}, \bar{d}_{n-\mu}\bar{d}_{n-\mu-1}, \ldots, \bar{d}_{n-\mu} \ldots \bar{d}_2)),$$

and the conclusion follows. □

Definition 1. The $\bar{d}_i(t)$, $i = 2, 3, \ldots, n - \mu$, given above in Lemma 4 are called the *singularity factors* of $B(t, 1)$.

Lemma 5. *Let $D_k(t, u)$ be the determinant factor of $B(t, u)$ of order k. We have*

$$D_k(t, 1) = \bar{d}_{n-\mu}(t)^k \bar{d}_{n-\mu-1}(t)^{k-1} \cdots \bar{d}_{n-\mu-k+2}(t)^2 \bar{d}_{n-\mu-k+1}(t),$$

where $k = 1, 2, \ldots, n - \mu - 1$.

Proof. By Lemma 4, $B(t, 1)$ is equivalent to its Smith form. Therefore, $B(t, 1)$ and its Smith form have the same determinant factors. Hence, the conclusion follows. □

To account for the parameter values at infinity, we consider $B(1, u)$. Similarly, we have

Lemma 6. *There exist unique polynomials $\hat{d}_i(u)$, $i = 2, \ldots, n - \mu$, such that*

$$R_2(u)B(1, u)S_2(u) = \text{diag}(\hat{d}_{n-\mu}(u), \hat{d}_{n-\mu}(u)\hat{d}_{n-\mu-1}(u), \ldots, \hat{d}_{n-\mu}(u) \ldots \hat{d}_2(u)),$$

for some invertible matrices $R_2(u)$, $S_2(u)$. Furthermore, $\mathbf{P}(1, u_0)$ is a singular point of order r if and only if $\hat{d}_r(u_0) = 0$ and $\hat{d}_i(u_0) \neq 0$, $i = r + 1, \ldots, n - \mu$.

Lemma 7.

$$D_k(1, u) = \hat{d}_{n-\mu}(u)^k \hat{d}_{n-\mu-1}(u)^{k-1} \cdots \hat{d}_{n-\mu-k+2}(u)^2 \hat{d}_{n-\mu-k+1}(u),$$

where $k = 1, 2, \ldots, n - \mu - 1$.

From the above lemmas, we have

Theorem 4. *Let $d_i(t, u) = LCM(\bar{d}_i(t, u), \hat{d}_i(t, u))$, where $\bar{d}_i(t, u)$ and $\hat{d}_i(t, u)$ are the homogenized polynomials of $\bar{d}_i(t)$ and $\hat{d}_i(t)$ respectively. Then*

$$D_k(t, u) = d_{n-\mu}(t, u)^k d_{n-\mu-1}(t, u)^{k-1} \cdots d_{n-\mu-k+2}(t, u)^2 d_{n-\mu-k+1}(t, u), \tag{8}$$

$k = 1, 2, \ldots, n - \mu - 1$, and

$$\begin{aligned}\text{rank}(B(t, u)) = \text{rank}(\text{diag}(&d_{n-\mu}(t, u), d_{n-\mu}(t, u)d_{n-\mu-1}(t, u), \ldots, \\ &d_{n-\mu}(t, u) \ldots d_2(t, u))).\end{aligned} \tag{9}$$

Furthermore, $\mathbf{P}(t_0, u_0)$ is a singular point of order r if and only if $d_r(t_0, u_0) = 0$ and $d_i(t_0, u_0) \neq 0$, $i = r + 1, \ldots, n - \mu$.

Remark: Here $d_i(t, u)$, $i = 2, 3, \ldots, n - \mu$ are called the *singularity factors* of $B(t, u)$.

Proof. We first prove (8) by induction on k. When $k = 1$, by Lemmas 5 and 7, there exist nonnegative integers $\lambda_{n-\mu}$ and $\delta_{n-\mu}$ such that

$$D_1(t, u) = \bar{d}_{n-\mu}(t, u)u^{\lambda_{n-\mu}} = \hat{d}_{n-\mu}(t, u)t^{\delta_{n-\mu}}.$$

It is easy to see that $d_{n-\mu} = LCM(\bar{d}_{n-\mu}(t,u), \hat{d}_{n-\mu}(t,u)) = D_1(t,u)$. Thus (8) holds for $k = 1$.

Now suppose that the statement is true for $1 \le k < n-\mu-1$. That is, $d_i(t,u) = \bar{d}_i(t,u)u^{\lambda_i} = \hat{d}_i(t,u)t^{\delta_i}$ for some nonnegative integers λ_i and δ_i, $i = n-\mu, \ldots, n-\mu-k+1$, and (8) holds for $i = 1, 2, \ldots, k$. Again from Lemmas 5 and 7, there exist nonnegative integers $\bar{\lambda}_{n-\mu-k}$ and $\bar{\delta}_{n-\mu-k}$ such that

$$\begin{aligned} D_{k+1}(t,u) &= \bar{d}_{n-\mu}(t,u)^{k+1}\bar{d}_{n-\mu-1}(t,u)^k \cdots \bar{d}_{n-\mu-k+1}(t,u)^2\bar{d}_{n-\mu-k}(t,u)u^{\bar{\lambda}_{n-\mu-k}} \\ &= \hat{d}_{n-\mu}(t,u)^{k+1}\hat{d}_{n-\mu-1}(t,u)^k \cdots \hat{d}_{n-\mu-k+1}(t,u)^2\hat{d}_{n-\mu-k}(t,u)t^{\bar{\delta}_{n-\mu-k}}. \end{aligned}$$

Substituting $\bar{d}_i(t,u) = d_i(t,u)u^{-\lambda_i}$ and $\hat{d}_i(t,u) = d_i(t,u)t^{-\delta_i}$, $i = n-\mu, \ldots, n-\mu-k+1$, into the above equation yields

$$D_{k+1} = d_{n-\mu}(t,u)^{k+1}d_{n-\mu-1}(t,u)^k \cdots d_{n-\mu-k+1}(t,u)^2\bar{d}_{n-\mu-k}(t,u)u^{\lambda_{n-\mu-k}},$$

and

$$\bar{d}_{n-\mu-k}(t,u)u^{\lambda_{n-\mu-k}} = \hat{d}_{n-\mu-k}(t,u)t^{\delta_{n-\mu-k}},$$

where

$$\lambda_{n-\mu-k} = \bar{\lambda}_{n-\mu-k} - \sum_{i=2}^{k+1} i\lambda_{n-\mu-k+i-1} \ge 0$$

and

$$\delta_{n-\mu-k} = \bar{\delta}_{n-\mu-k} - \sum_{i=2}^{k+1} i\delta_{n-\mu-k+i-1} \ge 0.$$

It is easy to verify that $d_{n-\mu-k}(t,u) = \bar{d}_{n-\mu-k}(t,u)u^{\lambda_{n-\mu-k}} = \hat{d}_{n-\mu-k}(t,u)t^{\delta_{n-\mu-k}}$. Hence, Eq. (8) holds also for $k+1$. This proves (8).

Eq. (9) and the second part of the theorem follow from Lemmas 4 and 6 by noticing that $d_i(t,u) = \bar{d}_i(t,u)u^{\lambda_i} = \hat{d}_i(t,u)t^{\lambda_i}$. □

Corollary 4. *Let $h_r(t,u)$ be the product of all the inversion formulas of order r singular points of $\mathbf{P}(t,u)$. Then $h_r(t,u)|d_r(t,u)$.*

Proof. By Lemma 3 and (8),

$$h_r|d_{n-\mu}^{n-\mu-r+1} \cdots d_{r+1}^2 d_r.$$

Since h_r is the inversion formula of the singular points of order r, $GCD(h_r, d_i) = 1$ for $i > r$. Therefore $h_r|d_r$. □

The following degree count will be useful.

Lemma 8. $\deg(D_{n-\mu-1}) = (n-1)(n-2)$.

Proof. We just sketch the proof, since it is similar to the proof of Theorem 2 in Chionh and Sederberg (2001).

Consider the linear system $B(t,u)\mathbf{v} = \mathbf{0}$, where $\mathbf{v}$ is a column vector of dimension $n-\mu$. Since $\text{rank}(B(t,u)) = n-\mu-1$ except at a finite number of parameter values at singular points, the linear system has exactly one solution. Obviously, $\mathbf{v} = (u^{n-\mu-1}, u^{n-\mu-2}t, \ldots, t^{n-\mu-1})^T$ is a solution by the construction of the Bézout matrix, and for each i $(1 \le i \le n-\mu)$, $\mathbf{v} = (B_{i,1}(t,u), B_{i,2}(t,u), \ldots, B_{i,n-\mu}(t,u))^T$ is also a solution. Here $B_{i,j}(t,u)$ is the algebraic cofactor of the (i, j) element of $B(t,u)$. Thus

$$\frac{B_{i,1}(t,u)}{u^{n-\mu-1}} = \frac{B_{i,2}(t,u)}{u^{n-\mu-2}t} = \cdots = \frac{B_{i,n-\mu}(t,u)}{t^{n-\mu-1}}.$$

Therefore $B_{ij}(t,u) = (-1)^{i+j}u^{2n-2-i-j}t^{i+j}D_{n-\mu-1}(t,u)$. The conclusion follows immediately by noticing that $\deg(B_{i,j}) = n(n-1)$. □

The next theorem enables us to tell whether a rational curve has any non-ordinary singular point with singular points in its infinitely near neighborhood . First recall $h_r(t,u)$ and $d_r(t,u)$ defined in Corollary 4 and Theorem 4.

Theorem 5. *$h_r(t,u) = d_r(t,u)$ for $r = 2, \ldots, n-\mu$, if and only if all the singular points of the rational curve $\mathbf{P}(t,u)$ do not contain singular points in their infinitely near neighborhoods.*

Proof. First consider sufficiency. Suppose that the rational curve $\mathbf{P}(t,u)$ has m_r singular points of order r. Since all the singular points do not have singular points in their infinitely near neighborhoods, from the genus formula (Walker, 1950), we have

$$\sum_{r=2}^{n-\mu} r(r-1) \times m_r = (n-1)(n-2).$$

On the other hand, by Theorem 4, Corollary 4 and Lemma 8, we have

$$\begin{aligned}(n-1)(n-2) &= \deg(D_{n-\mu-1}) = \sum_{r=2}^{n-\mu}(r-1)\deg(d_r) \\ &\ge \sum_{i=2}^{n-\mu}(r-1)\deg(h_r) \ge \sum_{i=2}^{n-\mu}(r-1) \times rm_r = (n-1)(n-2).\end{aligned}$$

Hence $\deg(d_r) = \deg(h_r)$, $r = 2, 3, \ldots, n-\mu$, i. e., $d_r = h_r$ if both are monic polynomials.

Now consider necessity. Suppose that $h_r(t,u) = d_r(t,u)$, for $r = 2, \ldots, n-\mu$. Then, $h_r(t,u)$ corresponds to all m_r singular points of order r, each of which is counted as $r(r-1)/2$ double points. Clearly, the degree of $h_r(t,u)$ is $\deg(h_r) = rm_r$. From $h_r(t,u) = d_r(t,u)$, it follows that

$$(n-1)(n-2) = \sum_{r=2}^{n-\mu}(r-1)\deg(d_r) = \sum_{r=2}^{n-\mu}(r-1)\deg(h_r) = \sum_{r=2}^{n-\mu}(r-1)rm_r.$$

Therefore,

$$\sum_{r=2}^{n-\mu} m_r\frac{(r-1)r}{2} = \frac{(n-1)(n-2)}{2}. \tag{10}$$

Hence, all the singular points do not have singular points in their infinitely near neighborhoods, for otherwise Eq. (10) does not hold (Walker, 1950). □

Theorems 4 and 5 lead to a modified version of the conjecture proposed in Chionh and Sederberg (2001).

Theorem 6. *Suppose that all the singular points of* $\mathbf{P}(t,u)$ *are ordinary, and* $\mathbf{P}(t_0,u_0)$ *is a singular point of order* r *such that* $t_0 : u_0$ *is a simple root of the inversion formula. Then* $t_0 : u_0$ *is a root of* $D_{n-\mu-1}(t,u)$ *with multiplicity* $r-1$.

Proof. Since the singular points of $\mathbf{P}(t,u)$ are ordinary, and $t_0 : u_0$ is a simple root of $d_r(t,u) = h_r(t,u)$, and $t_0 : u_0$ is not the root of $d_i(t,u)$, $i = 2,\ldots,n-\mu$ and $i \neq r$, by Theorem 4, $D_{n-m-1}(t,u)$ contains factor $(u_0t - t_0u)^{r-1}$, and the multiplicity of $u_0t - t_0u$ in $D_{n-\mu-1}(t,u)$ is exactly $r-1$. This completes the proof of the theorem. □

Remark 1. The conjecture in Chionh and Sederberg (2001) is based on the Bezoutian matrix derived from $c(t)x - a(t)$ and $c(t)y - b(t)$, and it is equivalent to Theorem 6 since the two Bezoutian matrices have essentially the same singularity factors.

Based on Theorem 5, we have a conjecture about the relationship between the singularity factors and the structure of non-ordinary singular points.

Conjecture 1. *Suppose that* $\mathbf{P}(t,u)$ *has* m_r *singular points of order* r. *Let* m_r^i *be the number of singular points that are the order* r *singular points in the neighborhood of order* i $(i \geq r)$ *singular points. Then*

$$d_r(t,u) = h_r(t,u)\prod_{i=r}^{n-\mu}\psi_r^i(t,u),$$

where $\psi_r^i(t,u)$ *is the factor corresponding to the* m_r^i *order* r *singular points in the neighborhood of order* $i\,(i \geq r)$ *singular points, and* $\deg(\psi_r^i(t,u)) = rm_r^i$.

Remark 2. Conjecture 1 depicts a clear structure tree of the non-ordinary singular points of a rational curve. If this conjecture is true, it would be very easy to decide whether a singular point is ordinary or not.

From the above results, to compute the singular points of $\mathbf{P}(t,u)$, we only need to study the singularity factors $d_i(t,u)$, $i = 2,\ldots,n-\mu$. However, the zeros of $d_r(t,u)$ may not contain only order r singular points, since it may contain a factor of other singularity factors $d_i(t,u)$, $(i > r)$, which corresponds to higher order singular points. To eliminate such ambiguity, we modify the singularity factors as follows.

For $l = n-\mu, n-\mu-1,\ldots,3$, eliminate the common factors of $d_l(t,u)$ and $d_i(t,u)$ from $d_i(t,u)$, $i = l-1,\ldots,2$. The modified singularity factors are denoted by $\tilde{d}_i(t,u)$, $i = 2,\ldots,n-\mu$. For example, suppose that $d_5(t,u) = t^2(t-u)^3$, $d_4(t,u) = (t+u)^2(t-2u)^2$, $d_3(t,u) = t(t-u)^2(t+2u)(t+3u)(t-3u)$ and $d_2(t,u) = t(t-u)(t+u)(t-2u)(t-5u)(t+6u)$. The modified singularity factors are then $\tilde{d}_5(t,u) = t^2(t-u)^3$, $\tilde{d}_4(t,u) = (t+u)^2(t-2u)^2$, $\tilde{d}_3(t,u) = (t+2u)(t+3u)(t-3u)$ and $\tilde{d}_2(t,u) = (t-5u)(t+6u)$.

GCD computation is needed to eliminate the common factor of two polynomials, say $f(t,u)$ and $g(t,u)$, from one polynomial, say $g(t,u)$.

Theorem 7. $\mathbf{P}(t_0,u_0)$ *is a singular point of order* r *if and only if* $\tilde{d}_r(t_0,u_0) = 0$.

Proof. Noticing that $\tilde{d}_r(t,u)$ is obtained by eliminating the common factors of $d_r(t,u)$ and $d_i(t,u)$, $i = r+1,\ldots,n-\mu$, the assertion follows directly from Theorem 4. □

Conjecture 2. *Suppose that $h_r(t,u) = \prod_{i=1}^{m_r} h_r^i(t,u)$, where $h_r^i(t,u)$ is the inversion formula for some order r singular point. Then*

$$\tilde{d}_r(t,u) = \prod_{i=1}^{m_r} h_r^i(t,u)^{l_i},$$

where l_i, $i = 1, 2, \ldots, m_r$ are positive integers.

Remark 3. The positive integer l_i in Conjecture 2 indicates that the singular point corresponding to $h_r^i(t,u)$ has $l_i - 1$ neighborhood singular points of order r.

Remark 4. By Theorem 2, there are no singular points of order between $\mu + 1$ and $n - \mu - 1$. Therefore, $\tilde{d}_i(t,u) = 1$ for $i = \mu + 1, \ldots, n - \mu - 1$.

Remark 5. For any $2 \le i \ne j \le n - \mu$, $\tilde{d}_i(t,u)$ and $\tilde{d}_j(t,u)$ do not have a common factor.

Remark 6. One may take the $n \times n$ Bézout matrix of polynomials $c(t,u)x - a(t,u)w$ and $c(t,u)y - b(t,u)w$ or the $\mu \times \mu$ matrix in Proposition 3 as the implicitization matrix $\hat{B}(x,y,w)$ and derive similar results in the above theorems. In the former case, the singularity factors will be $(1, \ldots, 1, d_{n-\mu}(t,u), \ldots, d_2(t,u))$, where the first μ singularity factors are all 1. In the latter case, the singularity factors are $(d_\mu(t,u), \ldots, d_2(t,u))$, which do not include the singularity factor of the order $n - \mu$ singular point (if any). One can find it by Theorem 2.

Remark 7. It is also possible to obtain information about the singular points of a rational parametric curve using the subresultant technique. As indicated in Abdeljaoued et al. (2004) that, for any two polynomials $P(t)$ and $Q(t)$,

$$\deg(\gcd(P,Q)) = r \leftrightarrow psc_0(P,Q) = \cdots = psc_{r-1}(P,Q) = 0, \quad psc_r(P,Q) \ne 0,$$

where $psc_i(P,Q)$, $i = 0, 1, \ldots$ is the principle subresultant (PS) sequence which can be computed from the minors of the Bezoutian matrix of P and Q. Applying the above result to singularity computation, one has to compute the subresultants of the μ-basis p and q, and substitute the parametric equation of the curve into the subresultants to get $psc_i(t,u)$. Then compute $g_r(t,u) := GCD(psc_0(P,Q), \ldots, psc_{r-1}(P,Q))$ and eliminate the common factors of $g_r(t,u)$ and $psc_r(P,Q)$ from $g_r(t,u)$ to get a polynomial $\hat{g}_r(t,u)$. Solving $\hat{g}_r(t,u) = 0$ should give the parametric values corresponding to all the order r singular points. It is an interesting question to find the relationship between $\hat{g}_r(t,u)$ and the singular factor $d_r(t,u)$.

4. Computing the singular points

Based on Theorem 7, a direct approach to computing the singular points of a rational plane curve is as follows. We first find the singularity factors $d_i(t,u)$ by computing the Smith form of the Bézout matrix $B(t,u)$, and then compute the modified singularity factors $\tilde{d}_i(t,u)$. The parameter values of the singular points of order i can be obtained by solving $\tilde{d}_i(t,u) = 0$, and the singular points are thus obtained simply by substituting the parameter values in $\mathbf{P}(t,u)$. Let us illustrate the process with an example.

Example 1. Consider a quartic curve:

$$a(t,u) = t^4 - 40t^3u + 40tu^3 + u^4,$$
$$b(t,u) = t^4 + 480t^2u^2 + u^4,$$
$$c(t,u) = t^4 + 40t^3u + 480t^2u^2 + 40tu^3 + u^4.$$

The μ-basis of the curve is computed as

$$p = (276x - 589y + 313w)t^2 + (-1434x - 1924y + 1878w)tu + (37x - 37y)u^2,$$
$$q = (37w - 37y)t^2 + (-1434x + 1388y - 1434w)tu + (-239x - 37y + 276w)u^2.$$

The implicitization matrix $\hat{B}(x, y, w)$ is the resultant matrix of p and q with respect to t and u, and the matrix $B(t, u)$ is obtained as

$$B(t, u) = d(t, u) \begin{pmatrix} u^2 & -tu \\ -tu & t^2 \end{pmatrix},$$

where

$$d(t, u) = -883200tu(37t^4 - 2868t^3u - 16656t^2u^2 + 2868tu^3 + 37u^4).$$

$B(t, u)$ has just one singularity factor $\phi(t, u)$ which gives all the parameter values of three double points. Solving $d(t, 1) = 0$ gives (10 digits accuracy):

$$t = 0, -5.583944445, -0.01205829975, 0.1790848763, 82.93043138.$$

Substituting these values of t and $u = 1$ into the parametric equation yields

$$(1, 1, 1), (0.8814197653, 1.821262166, 1), (0.8814197649, 1.821262166, 1),$$
$$(0.3335521852, 0.6892130165, 1), (0.3335521853, 0.6892130166, 1).$$

Note that the second and the third points correspond to the same double point, and the fourth and fifth points correspond to another double point. The two parameter values corresponding to the singular point $(1, 1, 1)$ are $t = 0$ and $u = 0$.

The above example reveals some problems in computing the singular points by directly solving the roots of the modified singularity factors. First, due to numerical errors, the number of computed singular points is bigger than expected. One has to regroup them such that each group corresponds to an actual singular point. If two singular points are too close, it will be difficult to properly regroup the singular points. Furthermore, it could happen that a singular point is computed as two distinct singular points. Second, to compute the singular points of order r, one has to solve the roots of a polynomial $\tilde{d}_r(t, u)$ whose degree is, in general, $r \times m_r$. Here m_r is the number of singular points of order r. Ideally, we would like to find the singular points of order r by solving the roots of a polynomial of degree m_r. In the following, we will propose a more efficient and numerically more stable algorithm to compute the singular points. Specifically, we will explore the following problems:

(1) Find a numerically more stable algorithm to compute the singular points.
(2) Compute the inversion formula for a singular point.
(3) Determine how many real singular points a rational plane curve segment has.
(4) Determine if a singular point is ordinary or not.

4.1. Compute singular points

For simplicity, we use $\tilde{d}(t, u)$ to denote the modified singularity factor which determines the singular points of some order r. Let m be the number of singular points of order r. To simplify the computation, we use the *reduced* (or *square-free*) polynomial $\tilde{d}_{red}(t, u)$ of $\tilde{d}(t, u)$ instead of $\tilde{d}(t, u)$. Here $\tilde{d}_{red}(t, u) = \tilde{d}(t, u)/GCD(\tilde{d}_t(t, u), \tilde{d}_u(t, u))$ is the polynomial that is striped off the repeated factors of $\tilde{d}(t, u)$. For example, for $\tilde{d}(t, u) = (t - 2u)^3(2t + u)^2(t - u)$, $\tilde{d}_{red}(t, u) = (t - 2u)(2t + u)(t - u)$.

The singular point (x, y, w) can be obtained by solving the following system of equations

$$\tilde{d}_{red}(t, u) = 0, \qquad c(t, u)x - a(t, u)w = 0, \qquad c(t, u)y - b(t, u)w = 0. \tag{11}$$

Let

$$\begin{aligned} f(x, w) &:= \mathbf{Res}(\tilde{d}_{red}(t, u), c(t, u)x - a(t, u)w), \\ g(y, w) &:= \mathbf{Res}(\tilde{d}_{red}(t, u), c(t, u)y - b(t, u)w). \end{aligned} \tag{12}$$

Solving $f(x, w) = 0$ and $g(y, w) = 0$ should give the x-coordinates and y-coordinates of the singular points, respectively. Note that, since we are only interested in finding the coordinates of the singular points rather than their multiplicities, we only need to find the roots of the reduced polynomials $f_{red}(x, w)$ and $g_{red}(y, w)$ of $f(x, w)$ and $g(y, w)$. Thus, we only need to find the roots of polynomials whose degree is less than or equal to m.

Let $x_i : 1, 1 \le i \le r$ and $y_j : 1, 1 \le j \le s$ be the roots of $f_{red}(x, w) = 0$ and $g_{red}(y, w) = 0$ respectively (for brevity of description, we assume that the roots are all finite). For each point $(x_i, y_j, 1)$, $1 \le i \le r, 1 \le j \le s$, we need to check if it is a singular point on the curve. To do so, we substitute $(x_i, y_j, 1)$ into the equation $G(x, y, w) := \mathbf{Res}(\tilde{d}_{red}(t, u), \alpha(c(t, u)x - a(t, u)w) + \beta(c(t, u)y - b(t, u)w))$, where $\alpha \in [0, 1]$ and $\beta \in [0, 1]$ are two randomly chosen numbers. If $G(x_i, y_j, 1) = 0$, then $(x_i, y_j, 1)$ is a singular point; otherwise, it is not. To increase the reliability, this process can be tested several times. Note that, since x_i and y_j are approximately computed, even if $(x_i, y_j, 1)$ is a singular point, $G(x_i, y_j, 1)$ is not exactly zero. So the test criteria is that the absolute value of $G(x_i, y_j, 1)$ is less than some threshold. The knowledge that we have in total m singular points is also helpful when determining the singular points.

4.2. *Inversion formula for singular points*

Let (x_0, y_0, w_0) be a singular point of order r, and $h(x_0, y_0, w_0; t, u)$ be the inversion formula for (x_0, y_0, w_0). By Lemma 2, $h(x_0, y_0, w_0; t, u)$ is in fact the GCD of two polynomials $f(x_0, y_0, w_0; t, u) := \mathbf{p}(t, u) \cdot (x, y, w)$ and $g(x_0, y_0, w_0; t, u) := \mathbf{q}(t, u) \cdot (x_0, y_0, w_0)$, where $\mathbf{p}(t, u)$ and $\mathbf{q}(t, u)$ are the μ-basis of $\mathbf{P}(t, u)$. Thus $h(x_0, y_0, w_0; t, u)$ can be computed symbolically by the subresultants of $f(x_0, y_0, w_0; t, u)$ and $g(x_0, y_0, w_0; t, u)$ (with respect to t, u) as follows.

Let

$$f(x_0, y_0, w_0; t, u) := \sum_{i=0}^{\mu} f_i(x_0, y_0, w_0)t^i u^{\mu-i}$$

and

$$g(x_0, y_0, w_0; t, u) := \sum_{i=0}^{n-\mu} g_i(x_0, y_0, w_0)t^i u^{n-\mu-i}.$$

Let S denote the Sylvester matrix of f and g, with respect to t, u. The first $n - \mu$ rows of S are coefficients (with respect to t, u) of f, and the last μ rows are coefficients of g. Let S_{ij} be the submatrix of S by deleting the last j rows of the $n - \mu$ rows of f-coefficients, and the last j rows of the μ rows of g-coefficients, and the last $2j + 1$ columns except the column $n - i - j$.

For $0 \le j \le \mu$, define

$$S_j(x_0, y_0, w_0; t, u) := \sum_{i=0}^{j} \det(S_{ij}) t^i u^{j-i}. \tag{13}$$

S_j is called the jth *subresultant* of f and g (with respect to t). By the property of subresultants (Wang, 2004), we have

Theorem 8. *Let (x_0, y_0, w_0) be a singular point of order r ($2 \le r \le \mu$). Then the inversion formula of the singular point is $h_r(x_0, y_0, w_0; t, u) := S_r(x_0, y_0, w_0; t, u) = 0$.*

Remark 8. All singular points of the same order r share the same inversion formula $h_r = 0$.

One can also find the inversion formula for each specific singular point (x_0, y_0, w_0) by computing the GCD of $f(x_0, y_0, w_0; t, u)$ and $g(x_0, y_0, w_0; t, u)$. However, since (x_0, y_0, w_0) is not exactly computed, $f(x_0, y_0, w_0; t, u)$ and $g(x_0, y_0, w_0; t, u)$ generally do not have an exact GCD. Thus numerical algorithms for computing approximate GCD must be employed. Detailed discussion in this direction is beyond the scope of the present paper.

To compute the parameter values of a singular point (x_0, y_0, w_0), one just solves for $t : u$ from the equation $h_r(x_0, y_0, w_0; t, u) = 0$, which has degree at most r. However, since $h_r(x_0, y_0, w_0; t, u) = 0$ is an approximate inversion formula for a specific singular point, the solutions may not be accurate. Fortunately, there is a way to improve the accuracy of the solutions. In fact, since any parameter $t_0 : u_0$ corresponding to the singular point is a solution of $\tilde{d}_{red}(t, u) = 0$, we can use $t_0 : u_0$ as an initial value to refine the solution of $\tilde{d}_{red}(t, u) = 0$ numerically using the Newton–Raphson iteration.

4.3. Detecting singular points on a real curve segment

In shape classification and control, it is often required to determine if there exist any singular points on a real curve segment, which is defined as a mapping $P : I \to R^2$, where $I \subset R$ is an interval. Without loss of generality, we assume that $I = [0, 1]$. This problem can be solved based on the inversion formula for the singular points. Since we are only interested in a finite portion of the curve, we dehomogenize the corresponding polynomials and coordinates in this subsection. Here we make it clear that a singular point of the whole curve $\mathbf{P}(t)$, $t \in R \cup \{\infty\}$, may not be a singular point on the segment $\mathbf{P}(t)$, $t \in [0, 1]$, since it may happen that only one of the parameters corresponding to the singular point lies in [0, 1].

First we need to find the coordinates of all singular points and test each of them to see if they are on the given curve segment.

Let $h(x, y; t)$ be the inversion formula for a singular point (x, y) of order r. Then (x, y) is a singular point on the curve segment $\mathbf{P}(t)$, $t \in [0, 1]$, if and only if $h(x, y; t)$ has more than one root in [0, 1] or has a multiple root in [0, 1]. We will explore the two cases separately.

A necessary condition for $h(x, y; t)$ having a multiple root is that $dis(x, y) := \mathbf{Res}(h, h'_t, t) = 0$. Thus if $dis(x, y) \neq 0$, then $h(x, y; t)$ cannot have a multiple root. If $dis(x, y) = 0$, we need to check if the multiple root is in [0, 1] or not. To do so, we compute $GCD(h, h'_t)$ and use Sturm sequences to check if the GCD has a root in [0, 1] or not.

Now suppose that $h(x, y; t)$ does not have multiple roots in [0, 1]. To determine whether the singular point (x, y) is a singular point on the curve segment or not, we just have to check if $h(x, y; t)$ has more than one root in [0, 1] using Sturm sequences. In particular, if (x, y) is the only singular point of order r, we just have too check the number of roots of $\tilde{d}(t) = 0$ in [0, 1].

4.4. Determine if a singular point is ordinary or not

There are several ways to determine if a singular point is ordinary or not, based on the representations of the rational curve.

The first approach is solely based on the parametric equation of $\mathbf{P}(t, u)$. Suppose we have a singular point (x_0, y_0) of order r and let $t_1 : u_1, t_2 : u_2, \ldots, t_r : u_r$ be the parameter values corresponding to (x_0, y_0). Without loss of generality, we assume that $(x_0, y_0) = (0, 0)$. Obviously, if there are identical parameter values among $t_i : u_i$, $i = 1, 2, \ldots, r$, then we have identical tangents at the singular point. Hence the singular point is non-ordinary. In the following, we assume that $t_i : u_i$, $i = 1, 2, \ldots, r$ are distinct. To compute the tangent lines at the singular point, we compute the places of $\mathbf{P}(t, u)$ at the singular point by taking Taylor expansions of $(a(t, u)/c(t, u), b(t, u)/c(t, u))$ at $t_i : u_i$, $i = 1, 2, \ldots, r$ respectively. From the places, it is easy to compute that the tangent directions are $\big(a'(t_i, u_i), b'(t_i, u_i)\big)$, $i = 1, 2, \ldots, r$. If there are identical tangent directions, then the singular point is non-ordinary. Otherwise, it is ordinary.

The second approach is based on the implicit equation of $\mathbf{P}(t, u)$ which can be easily obtained by taking the resultant of the μ-basis. Let $\mathbf{P}_0 = (x_0, y_0)$ be an order r singular point of $\mathbf{P}(t, u)$, and $F(x, y) = 0$ be the implicit equation. Expand $F(x, y)$ at $x = x_0$ and $y = y_0$, we get

$$F(x, y) := \sum_{r \le i+j \le n} F_{ij}(x_0, y_0)(x - x_0)^i (y - y_0)^j,$$

where $F_{ij}(x_0, y_0)$ is a polynomial in x_0, y_0. The directions of the tangent lines of $\mathbf{P}(t, u)$ at $\mathbf{P}_0$ are defined by the roots of

$$F_r(x, y) := \sum_{i+j=r} F_{ij}(x_0, y_0)(x - x_0)^i (y - y_0)^j = 0.$$

Let

$$f(x, y; \alpha, \beta) := \sum_{i+j=r} F_{ij}(x, y)\alpha^i \beta^j$$

and

$$g(x, y) := \mathbf{Res}(f'_\alpha, f'_\beta; \alpha, \beta). \tag{14}$$

Then $\mathbf{P}_0$ is non-ordinary if and only if $f(x_0, y_0; \alpha, \beta)$ has multiple roots (with respect to α, β), and if and only if $g(x_0, y_0) = 0$.

The above approaches are based on the fact that the singular points and/or the corresponding parameter values are already computed. If the singular points and/or the corresponding parameter values are not computed, how can we determine if the curve $\mathbf{P}(t, u)$ has any singular point or not of a specific order? Furthermore, even if the singular points and/or the corresponding parameter values are available, since they are, in general, approximately computed, serious numerical computation problem arises. For example, suppose $\mathbf{P}(t, u)$ has identical tangents at some non-ordinary singular point. But due to numerical computation, one may obtain distinct tangents, and hence one wrongly concludes that the singular point is ordinary. Besides, numerical stability is another problem for the first approach.

In the following, we provide a method to determine if $\mathbf{P}(t, u)$ has any non-ordinary singular point of a specific order r with the help of singularity factors. Note that all the computation can be exactly executed if the coefficients of the rational curve are rational numbers.

Define

$$\phi(t,u) := g(a(t,u)/c(t,u), b(t,u)/c(t,u)) \cdot c(t,u)^l, \tag{15}$$

where l is the degree of polynomial g. Let $h(t,u)$ be the inversion formula of $\mathbf{P}_0$ and $t_0 : u_0$ be a zero of $h(t,u)$, then $\mathbf{P}_0 = \mathbf{P}(t_0, u_0)$ is non-ordinary if and only if $\phi(t_0, u_0) = 0$.

Let $\tilde{d}(t,u)$ be the modified singularity factor of order r, and $\psi(t,u) = GCD(\tilde{d}(t,u), \phi(t,u))$. If $\psi(t,u)) = 1$, then $\mathbf{P}(t,u)$ cannot have non-ordinary singular points of order r. Otherwise, $\mathbf{P}(t,u)$ must have non-ordinary singular points. The non-ordinary singular point $\mathbf{P}_0 = (x_0, y_0)$ is determined by the following system of equations

$$\psi(t,u) = 0, \qquad c(t,u)x_0 - a(t,u) = 0, \qquad c(t,u)y_0 - b(t,u) = 0. \tag{16}$$

Theorem 9. *Let $\tilde{d}(t,u)$ be the modified singularity factor of order r, and $\psi(t,u) = GCD(\tilde{d}(t,u), \phi(t,u))$. Here $\phi(t,u)$ is the polynomial defined in* (15). *Then $\mathbf{P}(t,u)$ does not have non-ordinary singular points of order r if and only if $\psi(t,u) = 1$. Furthermore, the non-ordinary singular points of order r can be obtained by solving the system of equations* (16).

4.5. Algorithm

Based on the results in the previous subsections, we can devise algorithms to compute the singular points on a parametric curve and determine if a parametric curve segment has any singular points. Non-ordinary singular points can also be detected.

Algorithm 1. SINGULAR-COMPUT
Input: A parametric curve $\mathbf{P}(t,u) = (a(t,u), b(t,u), c(t,u))$.
Output: All the singular points of $\mathbf{P}(t,u)$.
Procedure:

1. Compute the singularity factors and then the modified singularity factors $d_i(t,u)$ and $\tilde{d}_i(t,u)$, $i = 2, \ldots, n-\mu$, of $\mathbf{P}(t,u)$.
2. For each modified singularity factor $\tilde{d}(t,u)$, compute $f(x,w)$ and $g(y,w)$ from Eq. (12). Let $f_{red}(x,w)$ and $g_{red}(y,w)$ be the reduced polynomials of $f(x,w)$ and $g(y,w)$ respectively.
3. Solve $f_{red}(x,1) = 0$ and $g_{red}(y,1) = 0$. Let x_i, $1 \le i \le m$ and y_j, $1 \le j \le m$ be the solutions respectively.
4. Choose some threshold $\epsilon > 0$ and let

   $$G(x,y) := \mathbf{Res}(\tilde{d}_{red}(t,u), \alpha(c(t,u)x - a(t,u)) + \beta(c(t,u)y - b(t,u)))$$

   be a randomly generated polynomial with α and β being two random numbers in [0, 1]. For each pair of combinations (x_i, y_j), $1 \le i, j \le m$, check if $|G(x_i, y_j)| < \epsilon$. If yes, then (x_i, y_j) is a singular point of order r, and output the singular point. To enhance the reliability, this step can be repeated several times with different random combinations (α, β).

Algorithm 2. SINGULAR-DETECT
Input: A parametric curve segment $\mathbf{P}(t) = (a(t), b(t), c(t))$, $0 \le t \le 1$.
Output: The singular points on the curve segment.
Procedure:

1. Compute the singularity factors and modified singularity factors $d_i(t)$ and $\tilde{d}_i(t)$, $i = 2, \ldots, n-\mu$.

2. For each modified singularity factor $\tilde{d}(t)$ (which corresponds to m singular points of order r), check the number of roots of $\tilde{d}(t)$ in [0, 1] using Sturm sequences. If $\tilde{d}(t) = 0$ has a simple root or no root in [0, 1], then the curve segment does not have a singular point of order r. Otherwise, if $m = 1$, then the curve segment has one singular point of order r; else, i.e., if $m > 1$, go to the next step.
3. Compute the singular points of order r of $\mathbf{P}(t)$ using the algorithm SINGULAR-COMPUT.
4. Compute the inversion formula $h(x, y; t) = 0$ of the singular points by Theorem 8.
5. For each singular point (x, y) of order r, check the number of zeros of $h(x, y; t) = 0$ again using Sturm sequence. If $h(x, y; t) = 0$ has a simple root or no root in [0, 1], then (x, y) is not a singular point on the curve segment; otherwise, it is.

Algorithm 3. NONSINGULAR-DETECT
Input: A rational parametric curve $\mathbf{P}(t, u) = (a(t, u), b(t, u), c(t, u))$.
Output: Determine if $\mathbf{P}(t, u)$ has any non-ordinary singular points of order r. If yes, compute the non-ordinary singular points.
Procedure:

1. Compute the singularity factors and then the modified singularity factors $d(t, u)$ and $\tilde{d}(t, u)$ of order r.
2. Compute the implicit equation $F(x, y, w) = 0$ of $\mathbf{P}(t, u)$ and the polynomial $\phi(t, u)$ as defined in (15).
3. Compute $\psi(t, u) = GCD(\phi(t, u), \tilde{d}(t, u))$. If $\psi(t, u) = 1$, then $\mathbf{P}(t, u)$ does not have any non-ordinary singular points. Otherwise, $\mathbf{P}(t, u)$ has non-ordinary singular points which can be solved from (16).

4.6. Examples

In this subsection, we provide several examples to illustrate the algorithms for computing and detecting the singular points of a planar rational curve.

Example 2. Consider again the curve in Example 1. As computed in Example 1, $B(t, u)$ has only one singularity factor

$$d(t, u) = -883200tu(37t^4 - 2868t^3u - 16656t^2u^2 + 2868tu^3 + 37u^4),$$

which gives all the parameter values of three double points. By (12), one computes

$$f_{red}(x) = (x - 1)(128190757x^2 - 155748174x + 37688017),$$
$$g_{red}(y) = (y - 1)(536363y^2 - 1346526y + 673263).$$

The roots of $f_{red}(x) = 0$ and $g_{red}(y) = 0$ give the x-coordinates and y-coordinates of the singular points respectively. It is easy to find that the three singular points are

$$(1, 1, 1), \qquad \left(\frac{325833}{536363} \pm \frac{370}{128190757}\sqrt{9007436063}, \frac{673263}{536363} \pm \frac{1110}{536363}\sqrt{74807}, 1\right).$$

The inversion formulae for the singular points are given by $p(x, y; t, u) = 0$ or $q(x, y; t, u) = 0$. For example, for $(x, y, w) = (1, 1, 1)$, the inversion formula is $tu = 0$.

To determine whether the segment $\mathbf{P}(t)$, $t \in [0, 1]$ has any singular points, we first use Sturm sequences to find that $\phi(t)$ has two simple roots in [0, 1]. We then conclude there might exist singular points in [0, 1].

For the point $(x, y) = (1, 1)$, $p(x, y; t) = t = 0$ has only one root in [0, 1]. Therefore, $(x, y) = (1, 1)$ is not a singular point on the curve segment $\mathbf{P}(t)$, $t \in [0, 1]$. Similarly, neither are the other two singular points on the curve segment. Hence, there is no singular point on the curve segment.

To determine whether $(x, y) = (1, 1)$ is ordinary or not, we expand $(a(t, u)/c(t, u), b(t, u)/c(t, u))$ at $(t, u) = (0, 1)$ and $(t, u) = (1, 0)$ respectively to get two tangent directions $(0, -40)$ and $(-80, -40)$ of the singular point. Thus $(x, y) = (1, 1)$ is an ordinary double point.

Example 3. The following example is from Cox et al. (1998). Let $\mathbf{P}(t, u)$ be given by

$$a(t, u) = t^6 + t^3u^3 + t^2u^4,$$
$$b(t, u) = t^6 - t^4u^2 - t^2u^4,$$
$$c(t, u) = t^6 + t^5u + t^4u^2 - tu^5 - u^6.$$

The μ-basis of $\mathbf{P}(t, u)$ is

$$p = (x - w)t^2 + ytu + yu^2,$$
$$q = (7y - 7w)t^4 + (8x + 7y - 8w)t^3u + (15y + 7w)t^2u^2 + 8ytu^3 + 7xu^4.$$

Now, since p is an axial moving line with the axis being $(1, 0, 1)$, $\mathbf{P}(t, u)$ has an order four singular point $(1, 0, 1)$. The inversion formula for the singular point is $q(1, 0, 1; t, u) = -7(t^4 - t^2u^2 - u^4) = 0$.

To compute the other singular points, we employ the matrix in Proposition 3 and obtain the singularity factor for $B(t, u)$ as

$$d(t, u) = 7t^2(2t^6 + 3t^5u + 3t^4u^2 + t^3u^3 + 3t^2u^4 + 3tu^5 + u^6),$$

whose roots are the parameter values of the four double points. From (12), one obtains

$$f_{red}(x) = x(3x^3 - 15x^2 - 3x - 1), \qquad g_{red}(y) = y(3y^3 - 21y^2 - 8).$$

The four double points are then found to be

$$(0, 0, 1), (5.204449622, 7.053597847, 1),$$
$$(-0.1022248111 - 0.2315120875i, -0.02679892333 + 0.6142796379i, 1),$$
$$(-0.1022248111 + 0.2315120875i, -0.02679892333 - 0.6142796379i, 1).$$

The inversion formula for these double points is $p(x_0, y_0, w_0; t, u) = 0$. Since $\phi(t, u)$ does not have a real root in $(0, \infty)$, $\mathbf{P}(t, u)$ does not have any singular point in $(0, \infty)$.

To determine whether the singular point $(x, y) = (1, 0)$ is ordinary or not, we compute the implicit equation $F(x, y) = 0$ of $\mathbf{P}(t, u)$ and expand $F(x, y)$ at $x = 1, y = 0$ to get its lowest degree terms: $F_4(x, y) = (x-1)^4 - 2(x-1)^3y + 6(x-1)y^3 - y^4$. The corresponding polynomial $f(\alpha, \beta) = \alpha^4 - 2\alpha^3\beta + 6\alpha\beta^3 - \beta^4$. Since $\mathit{resultant}(f'_\alpha, f'_\beta; \alpha, \beta) \neq 0$, $f(\alpha, \beta)$ does not have multiple roots, and hence the singular point $(x, y) = (1, 0)$ is ordinary.

To determine if $\mathbf{P}(t, u)$ has non-ordinary double points, we compute $\phi(t, u) = 8t^2(t^4 - t^2u^2 - u^4)^6 g(t, u)$ and $GCD(d(t, u), \phi(t, u)) = t^2$, where $g(t, u)$ is a polynomial of degree 30. Thus we know that $\mathbf{P}(t, u)$ has a non-ordinary double point $\mathbf{P}(0, 1) = (0, 0, 1)$.

Example 4. In this example, we consider a curve of relatively high degree. Let

$$a(t, u) = 3t^8 - 4t^7u + 5t^6u^2 - 4t^5u^3 + 4t^4u^4 - 4t^3u^5 - 3t^2u^6 + 5tu^7 - u^8,$$
$$b(t, u) = -t^8 - 4t^7u + 4t^6u^2 + 3t^5u^3 - 5t^4u^4 + 9t^2u^6 - 10tu^7 + 3u^8,$$
$$c(t, u) = 2t^8 - t^5u^3 - t^2u^6 + u^8.$$

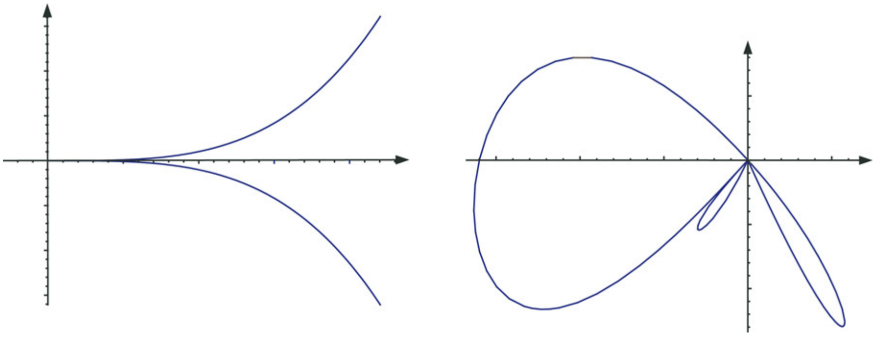


Fig. 1. Left: illustration of $x^7 - y^2 = 0$ in Example 5; Right: the rational curve in Example 6.

The μ-basis of $\mathbf{P}(t, u)$ is

$$\begin{aligned} p &= (126x + 52y - 163w)t^4 + (-6x + 10y + 370w)t^3u + (36x + 16y - 457w)t^2u^2 \\ &\quad + (61x - 14y + 93w)tu^3 + (6x + 2y)u^4, \\ q &= (10y + 5w)t^4 + (-384x - 158y + 517w)t^3u + (36x - 26y - 1171w)t^2u^2 \\ &\quad + (-107x - 56y + 1416w)tu^3 + (-183x + 44y - 315w)u^4. \end{aligned}$$

The singularity factors of matrix $B(t, u)$ are

$$d_2(t, u), \qquad d_3(t, u) = (t - u)^2(t - u), \qquad d_4(t, u) = 1,$$

where $d_3(t, u)$ corresponds to a triple point $(1, -1, 1)$, while $d_2(t, u)$ is a polynomial of degree 36, corresponding to 18 double points. Computing the subresultants of p and q yields an inversion formula for the double points

$$h(x, y, w; t) := A_2(x, y, w)t^2 + A_1(x, y, w)t + A_0(x, y, w) = 0,$$

where $A_i(x, y, w)$, $i = 0, 1, 2$, are degree five polynomials. The double points can be found by solving two degree 18 polynomials $f_{red}(x) = 0$ and $g_{red}(y) = 0$. Note that if we use the algorithm in Chionh and Sederberg (2001) to compute the double points using 10 digits accuracy, some of the double points lose several digits accuracy. For example, $(-0.06509039827, -1.566956501, 1)$ is a double point computed by our algorithm. However, the algorithm in Chionh and Sederberg (2001) computes two different points: $(-0.06509040134 \pm 0.1176641135 \times 10^{-9}i, -1.566956501 \pm 0.2825182231 \times 10^{-9}i, 1)$, which have only 7 digits accuracy. One can use Sturm sequences to show that $d_2(t, u) = 0$ does not have real roots. Hence, $\mathbf{P}(t, u)$ has only one real singular point $(1, -1, 1)$.

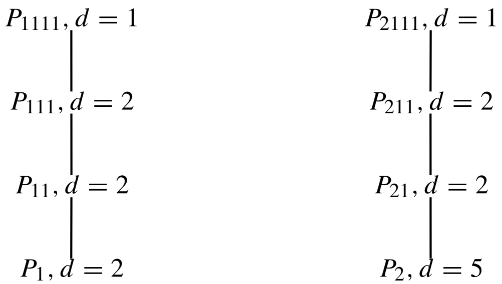
Next we use two more examples to demonstrate that the two conjectures Conjectures 1 and 2 proposed in the last section do hold in these two examples.

Example 5. Consider the rational curve as illustrated in Fig. 1

$$x(t) = t^2(t - 1)^5, \qquad y(t) = t^7, \qquad w(t) = (t - 1)^7.$$

Clearly the curve has the implicit equation $x^7 - y^2w^5 = 0$. It has two singular points—a non-ordinary double point $P_1 = (0, 0, 1)$ corresponding to $(t, u) = (0, 1)$, and a 5-fold point at $P_2 = (0, 1, 0)$ corresponding to $(t, u) = (1, 1)$.

By analyzing the singularities of the curve (see Abhyankar (1990, Lecture 19)), we have the following singularity trees:



The double point P_1 has a double point in its first neighborhood and a double point in its second neighborhood. The 5-fold point P_2 has also a double point in its first neighborhood and a double point in its second neighborhood.

Now let us check whether the Conjectures 1 and 2 are true or not for this example. The μ-basis of the rational curve is computed as

$$\begin{aligned} p &= (t^2 - 2tu + u^2)x - t^2 w, \\ q &= (6t^3u^2 - 9t^2u^3 + 5tu^4 - u^5)x - (t^5 - 5t^4u + 10t^3u^2 - 10t^2u^3 + 5tu^4 - u^5)y \\ &\quad + (t^5 + 2t^4u - 3t^3u^2 + t^2u^3)w. \end{aligned}$$

The Smith form of the Bézout resultant matrix of p and q is $\mathrm{diag}((t-u)^5, (t-u)^5, (t-u)^5, t^6(t-u)^9, 0)$.

From the Smith norm, we obtain $d_5(t,u) = (t-u)^5$, $d_4(t,u) = d_3(t,u) = 1$, and $d_2(t,u) = t^6(t-u)^4$. Obviously, the rational curve has two singular points—a double point $P_1 = (0,0,1)$ (which corresponds to the parameter value $(t,u) = (0,1)$) and a 5-fold point $P_2 = (0,1,0)$ (which corresponds to the parameter value $(t,u) = (1,1)$). The inversion formulas of the two singular points are $h_2(t,u) = t^2$ and $h_5(t,u) = (t-u)^5$, respectively.

Based on the singularity tree, we have $\psi_2^5(t,u) = (t-u)^4$, $\psi_2^2(t,u) = (t^2)^2 = t^4$. On the other hand, our computation yields $d_5(t,u) = h_5(t,u)$, $d_2(t,u) = h_2(t,u)\psi_2^5(t,u)\psi_2^2(t,u)$ and $\tilde{d}_2(t,u) = t^6 = h_2(t,u)^3$. Hence, both Conjectures 1 and 2 are true for this example.

Example 6. Consider the rational curve as illustrated in Fig. 1

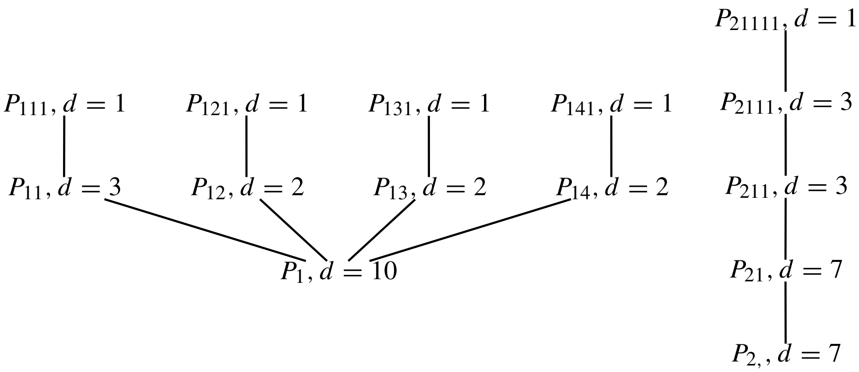
$$\begin{aligned} x(t,u) &= t^7(5t-u)^2(2t-u)^2(4t-u)^3(t-u)^3, \\ y(t,u) &= -(3t-u)(1048t^6 - 3384t^5u + 3912t^4u^2 - 2196t^3u^3 + 633t^2u^4 \\ &\qquad - 90tu^5 + 5u^6)(2t-u)^2(4t-u)^3(5t-u)^2(t-u)^3, \\ z(t,u) &= t^{17}. \end{aligned}$$

Using the symbolic software **Maple**, we first implicitize the rational curve, and then compute the singularities for the implicit curve to obtain the following singular points of the curve,

- $P_1 = (0,0,1)$: multiplicity: 10; delta invariant: 51; the number of local branches: 4;
- $P_2 = (0,1,0)$: multiplicity: 7; delta invariant: 48; the number of local branches: 1;
- other 21 ordinary double points, each having delta invariant equal to 1.

Here the delta invariant is the equivalent number of double points of the singular point, including those in its infinite neighborhood, should be counted for. For a rational curve of degree n, the delta invariants of all the singular points sum up to $(n-1)(n-2)/2$ (Walker, 1950).

The singularity trees are as follows:



The μ-basis of the rational curve is

$$\begin{aligned}
p &= yt^7 + (3144t^7 - 11200t^6u + 15120t^5u^2 - 10500t^4u^3 + 4095t^3u^4 - 903t^2u^5 \\
&\quad + 105tu^6 - 5u^7)x, \\
q &= (4610013111017257007205945645775441787 60t^9 \\
&\quad - 10618799367833093899157377430382413474 08t^8u \\
&\quad + 13041009149186188572669852108809632154 40t^7u^2 \\
&\quad - 96344657431182011332094677838234991734 0t^6u^3 \\
&\quad + 45304681427285650218834792274287270937 5t^5u^4 \\
&\quad - 13867739879508862103517269141680023028 5t^4u^5 \\
&\quad + 27519175579414801620922982185410211771t^3u^6 \\
&\quad - 3413749009440773749339485419802971315t^2u^7 \\
&\quad + 240614259752476291340326794064525710tu^8 \\
&\quad - 7358652103980467780171952529147300u^9)x \\
&\quad - t^7(25606870338581626368379674846725000t^3 \\
&\quad - 55408512503057088229879836607577665t^2u \\
&\quad + 17216513113777293591343158190486482tu^2 \\
&\quad - 1471730420796093556034390505829460u^3)y \\
&\quad - 80508000344500633302185697718103400000(5t-u)^2 \\
&\quad \times (2t-u)^2(4t-u)^3(t-u)^3w.
\end{aligned}$$

The Smith form of the Bézout matrix is diag$(\theta_1(t,u), \theta_1(t,u), \theta_1(t,u), t^{14}\theta_1(t,u), t^{14}\theta_1(t,u), t^{14}\theta_1(t,u), t^{14}\theta_1(t,u), t^{20}\theta_1(t,u)(4t-u)^3, t^{20}\theta_2(t,u)\theta_3(t,u), 0)$, where $\theta_1(t,u) = (5t-u)^2(4t-u)^3(2t-u)^2(t-u)^3$, $\theta_2(t,u) = (5t-u)^4(4t-u)^6(2t-u)^4(t-u)^5$, and $\theta_3(t,u)$ is a 42-degree polynomial.

From the Smith form, we obtain $d_{10}(t,u) = (5t-u)^2(2t-u)^2(4t-u)^3(t-u)^3$, $d_7(t,u) = t^{14}$, $d_3(t,u) = (4t-u)^3t^6$, $d_2(t,u) = \theta_3(t,u)(t-u)^2(2t-u)^2(5t-u)^2$ and the other singularity factors are all 1. Obviously, $P_1 = (0,0,1)$ is a 10-fold singular point and $P_2 = (0,1,0)$ is a 7-fold singular point. The inversion formulas are $h_{10}(t,u) = d_{10}(t,u)$ and $h_7(t,u) = t^7$, respectively. Based on the singularity tree, we have $\psi_3^{10}(t,u) = (4t-h)^3$, $\psi_2^{10}(t,u) = (t-u)^2(2t-u)^2(5t-u)^2$, $\psi_7^7(t,u) = t^7$ and $\psi_3^7(t,u) = t^6$. On the other hand, our computation yields $d_{10}(t,u) = h_{10}(t,u)$, $d_7(t,u) = h_7(t,u)\psi_7^7(t,u)$, $d_3(t,u) = \psi_3^{10}\psi_3^7$

and $d_2(t, u) = \theta_3(t, u)\psi_2^{10}(t, u)$. Hence, Conjecture 1 is true for this example. Similarly, we can easily verify that Conjecture 2 is also true for this example.

5. Singular points of low degree curves

For low degree curves such as cubic or quartic curves, there exists a relatively simple treatment to the singular points of the curves based on the main results from previous sections.

5.1. *Cubic curves*

For a cubic curve $\mathbf{P}(t)$, the μ-basis takes the form

$$p = p_1(x, y)t + p_0(x, y), \qquad q = q_2(x, y)t^2 + q_1(x, y)t + q_0(x, y),$$

where p_i, q_i are linear functions in x, y. Thus p is an axial moving line whose axis (x_0, y_0) (the intersection of $p_1 = 0$ and $p_0 = 0$) is the double point on the cubic curve. The inversion formula is provided by $q(x_0, y_0; t) = 0$, from which the existence of the singular point on a curve segment $\mathbf{P}(t)$, $t \in [0, 1]$ can easily be detected. The conditions are:

$$(h_2 + h_1 + h_0)(h_1 + 2h_0) \le 0, \qquad (h_2 + h_1 + h_0)h_0 \ge 0, \qquad h_1^2 - 4h_2h_0 \ge 0, \quad (17)$$

where $h_2 + h_1 + h_0 \ne 0$, $h_2 = q_2(x_0, y_0, w_0)$, $h_1 = q_1(x_0, y_0, w_0)$ and $h_0 = q_0(x_0, y_0, w_0)$.

5.2. *Quartic curves*

For a quartic curve $\mathbf{P}(t)$, there are two cases to be considered. In the first case, the μ-basis p and q have degree one and three, respectively, since $\mu = 1$. Therefore, the quartic curve has one singular point—a triple point (x_0, y_0) which can be obtained by intersecting any two lines of the moving p, and $q(x_0, y_0; t) = 0$ gives the inversion formula. Write $q(x_0, y_0, w_0; t, 1) := h(t) = h_3t^3+t_2t^2+h_1t+h_0$. Then a necessary and sufficient condition for the triple singular point to be on the curve segment $\mathbf{P}(t)$ ($t \in [0, 1]$) is that one of the following conditions holds

$$\begin{array}{lllll} \text{(i)} & g_0 = 0, & g_2g_1 \le 0, & g_2^2 - 4g_1g_3 \ge 0; & \\ \text{(ii)} & g_0g_1 < 0, & \Delta_1 \ge 0, & g_0\Delta_2 > 0, & \Delta_3 = 0; \\ \text{(iii)} & g_0g_1 < 0, & g_0g_3 > 0, & \Delta_2 > 0 & \Delta_3 < 0, \end{array} \quad (18)$$

where $g_0 = h_0$, $g_1 = h_1 + 3h_0$, $g_2 = h_2 + 2h_1 + 3h_0$, $g_3 = h_3 + h_2 + h_1 + h_0 \ne 0$, and

$$\Delta_1 = 3g_3g_1 - g_2^2, \qquad \Delta_2 = 9g_0g_3 - g_1g_2,$$
$$\Delta_3 = 4g_1^3g_3 - g_1^2g_2^2 - 18g_0g_1g_2g_3 + 4g_0g_2^3 + 27g_0^2g_3^2.$$

For the second case, $p = p_2t^2 + p_1t + p_0$ and $q = q_2t^2 + q_1t + q_0$ are both of degree two, since $\mu = 2$. In this case, $\mathbf{P}(t)$ has three double points, and the corresponding singularity factor is

$$\phi(t) = p_1(a, b, c)q_2(a, b, c) - p_2(a, b, c)q_1(a, b, c).$$

The x-coordinates and y-coordinates of the double points are the solutions of the cubic polynomials $f_{red}(x)$ and $g_{red}(y)$, respectively, where

$$f(x) = \mathbf{Res}(\phi(t, u), c(t, u)x - a(t, u)w; t, u),$$
$$g(y) = \mathbf{Res}(\phi(t, u), c(t, u)y - b(t, u)w; t, u).$$

The inversion formula is given by $p = 0$ or $q = 0$. Thus, one just needs to count the number of roots of a quadratic polynomial in the interval [0, 1] in order to detect any singular point on the curve segment.

5.3. *Monomial curves*

A degree n monomial curve is a parametric curve with a singular point of order $n - 1$ (which is the only singular point of the curve). For a monomial curve, the two elements p and q of the μ-basis are of degree one and $n - 1$ respectively. The singular point (x_0, y_0) is the intersection of any two lines of the moving line p. The inversion formula is given by $q(x_0, y_0; t) = 0$. One can use Sturm sequences to detect if the singular point is on a curve segment.

6. Conclusions

In this paper, we present a new approach to computing the singular points of a plane rational curve using the implicitization matrix derived from the μ-basis of the curve. It is shown that the singularity factors of the implicitization matrix provide all the information about the singular points, such as the parameter values corresponding to the singular points and their orders. Based on this result, an algorithm is presented to compute and detect the singular points. Our algorithm only requires to solve several univariate polynomial equations of relative low degrees, whereas previous methods either require to solve polynomial equations with two or three variables or to compute the zeros of a polynomial of much higher degree. Thus our algorithm is not only more efficient but also numerically more robust. Furthermore, inversion formulae for the singular points are derived, a method is presented to determine if a singular point is ordinary or not, and a conjecture made in Chionh and Sederberg (2001) regarding the multiplicity of singular points is proved.

There are several related problems for further research. In order to detect whether a given rational curve segment contains any singular point, we now first need, in general, to compute numerically all singular points of the whole curve and then find the inversion formulae of these singular points. This introduces numerical inaccuracy in the process. It would be an interesting research problem to find the inversion formula and to detect if a curve segment contains any singular point without first having to compute the singular points.

Another problem is to prove our conjecture (Conjecture 1) about the relationship between the singularity factors of the Bézout matrix and the singular points in infinitely near neighborhoods of other singular points.

Acknowledgements

The first author is supported by the National Key Basic Research Project of China (No. 2004CB318000), the Outstanding Youth Grant of NSF of China (No.60225002), the Doctoral Program of MOE of China and the 111 Project (No. B07033).

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